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On pointwise inner automorphisms of nilpotent groups of class2

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Abstract

An automorphism θ of a group G is pointwise inner if $\theta(x)$ is conjugate to x for any $x \in G$. The set of all pointwise inner automorphisms of group G, denoted by $\mathrm{Aut}_{\mathrm{pwi}}(G)$ form a subgroups of $\mathrm{Aut}(G)$ containing $\mathrm{Inn}(G)$. In this paper, we find a necessary and sufficient condition in certain finitely generated nilpotent groups of class 2 for which $\mathrm{Aut}_{\mathrm{pwi}}(G) \simeq \mathrm{Inn}(G)$. We also prove that in a nilpotent group of class 2 with cyclic commutator subgroup $\mathrm{Aut}_{\mathrm{pwi}}(G) \simeq \mathrm{Inn}(G)$ and the quotient $\mathrm{Aut}_{\mathrm{pwi}}(G)/\mathrm{Inn}(G)$ is torsion. In particular if G' is a finite cyclic group then $\mathrm{Aut}_{\mathrm{pwi}}(G) = \mathrm{Inn}(G)$.

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Introduction

By definition, a pointwise inner automorphism of a group G is an automorphism $\theta\colon G \to G$ such that t and $\theta(t)$ are conjugate for any $t \in G$. This notion appears in the famous book of Burnside [1, Note B, p 463]. Denote by $\mathrm{Aut}_{\mathrm{pwi}}(G)$ the set of all pointwise inner automorphisms of G.

Obviously, $\mathrm{Aut}_{\mathrm{pwi}}(G)$ contains $\mathrm{Inn}(G)$, the group of all inner automorphisms of G. These groups can coincide, for instance when G is $S_n, A_n, SL_n(D)$ and $GL_n(D)$ where D is an Euclidean domain (see [7], [10], [11]).

By a result of Grossman [5], it turns out that $\mathrm{Aut}_{\mathrm{pwi}}(G) = \mathrm{Inn}(G)$ when G is a free group. Endimioni in [4] proved that this property remains true in a free nilpotent group.

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Also Yadav in [12] gave a sufficient condition for a finite p-group G of nilpotent class 2 to be such that $\mathrm{Aut}_{pwi}(G) = \mathrm{Inn}(G)$. But the equality does not hold in general.

In fact, in 1911, Burnside posed the following question: Does there exist any finite group G such that G has a non-inner and pointwise inner automorphism? In 1913, Burnside himself gave an affirmative answer to this question [3]. Indeed, there are many examples of groups admitting a pointwise inner automorphism which is not inner (see, for instance [3], [4], [8], [9], [12] where these groups are besides nilpotent).

Segal also gave a subtle example. He constructed a finitely generated torsion-free nilpotent group G, in which $\mathrm{Aut}_{pwi}(G)/\mathrm{Inn}(G)$ contains an element of infinite order (see [9]).

In this paper we study the pointwise inner automorphisms of a finitely generated nilpotent group of class 2 with cyclic commutator subgroup.

We introduce the following definition:

Definition. Let G be a finitely generated nilpotent group of class 2. Then $G/Z(G)$ is finitely generated abelian group and thus $G/Z(G) = \langle x_1Z(G)\rangle \times \ldots \times \langle x_kZ(G)\rangle$ for some $x_1, \ldots, x_k \in G$. The group G is called d-group if the following distributive law holds in G,

$$[x_1^{\alpha_1} \ldots x_k^{\alpha_k}, G] = [x_1, G]^{\alpha_1} \ldots [x_k, G]^{\alpha_k}$$

where $\alpha_i \in \mathbb{Z}$ and $1 \leq i \leq k$.

Let G be a 2-generator nilpotent group of class 2. It is straightforward to show that G is a d-group.

To give an example of an infinite d-group, consider the group G with the following presentation

$$G = \langle x_1, x_2, x_3, x_4, x : [x_i, x_j] = x^{m_{ij}}, [x_i, x] = 1; 1 \leq i \leq 4 \text{ and} i < j \rangle,$$

where $m_{ii+1} = 1$ for all $1 \leq i < 4$ and $m_{ij} = 0$ for all $i + 1 < j$. Then $G' = Z(G) = \langle x \rangle \simeq \mathbb{Z}$ and $G/Z(G) = \langle \overline{x_1}, \overline{x_2}, \overline{x_3}, \overline{x_4} \rangle \simeq \mathbb{Z}^4$. A quick calculation shows that

$$[x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_4}, G] = [x_1, G]^{\alpha_1} [x_2, G]^{\alpha_2} [x_3, G]^{\alpha_3} [x_4, G]^{\alpha_4} = \langle x^{\alpha} \rangle,$$

Where $\alpha_i \in \mathbb{Z}$ for all $1 \leq i \leq 4$ and $\alpha = \gcd(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Therefore G is an infinite d-group.

Now we give a nilpotent group G of class 2 which is not a d-group.

Let G be a free nilpotent group of class 2 on 4 generators a_1, a_2, a_3 and a_4. If $c_{ij} = [a_i, a_j]$ for $1 \leq i < j \leq 4$, then the relations in G are $[c_{ij}, a_k] = 1$ for $1 \leq i < j \leq 4$ and $1 \leq k \leq 4$, and their consequences. Macdonald in [6] proved that $c_{13}c_{24}$ is not a commutator. Therefore G is not a d-group.

Theorem 1. Let G be a finitely generated nilpotent group of class 2 and

$$G/Z(G) = \langle \overline{x_1} \rangle \times \ldots \times \langle \overline{x_k} \rangle.$$

(i) There exists a monomorphism $\mathrm{Aut}_{\mathrm{pwi}}(G) \hookrightarrow \prod_{i=1}^{k} \mathrm{Hom}(\langle \overline{x_i} \rangle, [x_i, G])$.

(ii) If $[x_i, G]$ is cyclic for all $1 \leq i \leq k$, then there exists a monomorphism $\mathrm{Aut}_{\mathrm{pwi}}(G) \hookrightarrow \mathrm{Inn}(G)$.

In particular if G is a d-group of class 2 then the monomorphisms in (i) and (ii) are isomorphism. Furthermore $\mathrm{Aut}_{\mathrm{pwi}}(G) \simeq \mathrm{Inn}(G)$ if and only if $[x_i, G]$ is cyclic for all $1 \leq i \leq k$.

Notice that if G is a finite group then, as consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

In particular, we derive the following consequences of Theorem 1.

Corollary 1. Let G be a finitely generated nilpotent group of class 2 in which G' is cyclic, then $\mathrm{Aut}_{\mathrm{pwi}}(G) \simeq \mathrm{Inn}(G)$. In particular if G' is finite, then $\mathrm{Aut}_{\mathrm{pwi}}(G) = \mathrm{Inn}(G)$.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if G' is cyclic then $\mathrm{Aut}_{\mathrm{pwi}}(G) = \mathrm{Inn}(G)$. But we cannot hope for a similar conclusion when G is not finite. We will provide an example in the section 2. However, in a finitely generated nilpotent group of class 2, by Corollary 1 we have $\mathrm{Aut}_{\mathrm{pwi}}(G) \simeq \mathrm{Inn}(G)$. So the structure of $\mathrm{Aut}_{\mathrm{pwi}}(G)$ is determined.

Corollary 2. Let G be a finitely generated nilpotent group of class 2. If the commutator subgroup of G is cyclic, then $\mathrm{Aut}_{\mathrm{pwi}}(G)/\mathrm{Inn}(G)$ is torsion.

Let G be a group and N be a non-trivial proper normal subgroup of G. The pair

(G, N) is called a Camina pair if $xN \subseteq x^G$ for all $x \in G \backslash N$. A group G is called a Camina group if (G, G') is a Camina pair.

Clearly, if G is a Camina group of class 2 then it is a d-group. So, as an immediate consequence of Theorem 1, one readily gets the following corollary.

Corollary 3. Let G be a finitely generated nilpotent group of class 2. If G is a Camina group then $\mathrm{Aut}_{pwi}(G) \simeq \mathrm{Inn}(G)$ if and only if G' is cyclic. Particularly, if $G/Z(G)$ is finite, then $\mathrm{Aut}_{pwi}(G) = \mathrm{Inn}(G)$ if and only if G' is cyclic.

Preliminary results

Our notation is standard. Let G be a group, by C_m, G' and $Z(G)$, we denote the cyclic group of order m, the commutator subgroup and the center of G, respectively.

If $x, y \in G$, then x^y denotes the conjugate element $y^{-1}xy \in G$. For $x \in G$, x^G denotes the conjugacy class of x in G. The commutator of two elements $x, y \in G$ is defined by $[x, y] = x^{-1}y^{-1}xy$ and more generally, the left-normed commutator of n elements $x_1, \dots, x_n$ is defined inductively by

$$[x_1, \dots, x_{n-1}, x_n] = [x_1, \dots, x_{n-1}]^{-1}x_n^{-1}[x_1, \dots, x_{n-1}]x_n.$$

If $H \leq G$, $[x, H]$ denotes the set of all $[x, h]$ for $h \in H$, this is a subgroup of G when G is of class 2. For any group H and abelian group K, $\mathrm{Hom}(H, K)$ denotes the group of all homomorphisms from H to K. Also C^* is the set of all central automorphisms of G fixing $Z(G)$ elementwise.

Yadav in [12] shows that in a finite nilpotent group of class 2, there exists a monomorphism from $\mathrm{Aut}_{pwi}(G)$ into $\mathrm{Hom}(G/Z(G), G')$. It turns out that this result remains true when G is an infinite nilpotent group of class 2.

For that, let G be a nilpotent group (finite or infinite) of class 2. Let $\alpha \in \mathrm{Aut}_{pwi}(G)$. Then the map $g \mapsto g^{-1}\alpha(g)$ is a homomorphism from G into G'. This homomorphism sends $Z(G)$ to 1. So it induces a homomorphism $f_\alpha: G/Z(G) \to G'$, sending $\overline{g} = gZ(G)$ to $g^{-1}\alpha(g)$, for any $g \in G$. Define

$$\mathrm{Hom}_{pwi}(G/Z(G), G') = \{f \in \mathrm{Hom}\left(\frac{G}{Z(G)}, G'\right) : f(\overline{g}) \in [g, G] \text{ for all } g \in G\}.$$

To prove $\mathrm{Aut}_{pwi}(G) \simeq \mathrm{Hom}_{pwi}(G/Z(G), G')$, we use the following well-known result.

Lemma 1.1 Let N be a normal subgroup of a group G. Let θ be an endomorphism of G such that $\theta(N) \leq N$. Denote by $\overline{\theta}$ and θ_0 the endomorphisms induced by θ in G/N and N, respectively. If $\overline{\theta}$ and θ_0 are surjective (injective), then so is θ.

Proposition 1.2 Let G be a nilpotent group of class 2. Then the above map $\varphi: \alpha \mapsto f_\alpha$ is an isomorphism from $\mathrm{Aut_{pwi}}(G)$ into $\mathrm{Hom_{pwi}}(G/Z(G), G')$.

Proof. Since for any $\alpha \in \mathrm{Aut_{pwi}}(G)$, by the definition $f_\alpha \in \mathrm{Hom_{pwi}}(G/Z(G), G')$, φ is well defined. Let $\alpha_1, \alpha_2 \in \mathrm{Aut_{pwi}}(G)$ and $g \in G$. We have $\alpha_1(g^{-1}\alpha_2(g)) = g^{-1}\alpha_2(g)$, since $g^{-1}\alpha_2(g) \in G' \leq Z(G)$. This implies that

$$f_{\alpha_1\alpha_2}(\overline{g}) = g^{-1}\alpha_1(\alpha_2(g)) = g^{-1}\alpha_1(gg^{-1}\alpha_2(g))$$
$$= g^{-1}\alpha_1(g).\, g^{-1}\alpha_2(g) = f_{\alpha_1}(\overline{g}).\, f_{\alpha_2}(\overline{g}).$$

Hence φ is a homomorphism. Clearly, φ is injective. Now it suffices to show that φ is surjective.

Let f be any element of $\mathrm{Hom_{pwi}}(G/Z(G), G')$. By Lemma 1.1 a quick calculation shows that $\varphi(\alpha) = f$, where α is an element of $\mathrm{Aut_{pwi}}(G)$, sending $g \in G$ to $gf(gZ(G))$. Then we have $\mathrm{Aut_{pwi}}(G) \simeq \mathrm{Hom_{pwi}}(G/Z(G), G')$.

* Note that if G is a nilpotent group of class 2 then $\mathrm{Aut_{pwi}}(G) \simeq \mathrm{Hom_{pwi}}(G/Z(G), G')$. It is easy to see that in a Camina nilpotent group of class 2, $\mathrm{Hom_{pwi}}(G/Z(G), G') = \mathrm{Hom}(G/Z(G), G')$. Hence if G is a Camina group of class 2, then $\mathrm{Aut_{pwi}}(G) \simeq \mathrm{Hom}(G/Z(G), G')$.

The following well-known facts will be used repeatedly.

Lemma 1.3 Let A, B and C be abelian groups.

(i) $\mathrm{Hom}(A \times B, C) \simeq \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C)$.

(ii) $\mathrm{Hom}(A, B \times C) \simeq \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, C)$.

(iii) $\mathrm{Hom}(C_m, C_n) \simeq C_d$ where $d = \gcd(m, n)$.

(iv) $\mathrm{Hom}(\mathbb{Z}, A) \simeq A$.

(v) If A is torsion group and B is torsion-free group, then $\mathrm{Hom}(A, B) = 1$.

(vi) If $\gcd(|A|, |B|) \neq 1$, then $\mathrm{Hom}(A, B) \neq 1$.

Main Result

Let G be a finite abelian group. We denote by G_p, the p-primary component of G. Hence $G = \Pi_{p\in\pi(G)} G_p$ where $\pi(G)$ denotes the set of all primes p dividing $|G|$.

To prove Theorem 1, we need the following Lemma.

Lemma 2.1 ([1, Corollary 1.4]) Let A and B be two finite abelian groups and $\exp(A)|\exp(B)$. Then $\mathrm{Hom}(A,B) \simeq A$ if and only if $B \simeq C_m \times H$ in which $C_m \simeq \Pi_{p\in\pi(A)} B_p$ and $H \simeq \Pi_{p\notin\pi(A)} B_p$. In particular, if $\pi(A) = \pi(B)$ then this is equivalent to B is a cyclic group.

Let G be a finitely generated nilpotent group of class 2. Then $G/Z(G)$ is finitely generated abelian group and thus $G/Z(G) = \langle x_1 Z(G)\rangle \times\ldots\times \langle x_k Z(G)\rangle$ for some $x_1, \ldots, x_k \in G$.

Let $f \in \mathrm{Hom}_{pwi}(G/Z(G), G')$. So $f\big(gZ(G)\big) \in [g, G]$ for all $g \in G$. In particular, for all $1 \le i \le k$ we have $f(x_i Z(G)) \in [x_i, G]$. Now we prove Theorem 1.

Proof of Theorem 1.

(i) By Proposition 1.2, we have $\mathrm{Aut}_{pwi}(G) \simeq \mathrm{Hom}_{pwi}(G/Z(G), G')$. It suffices to show that there exists a monomorphism from $\mathrm{Hom}_{pwi}(G/Z(G), G')$ into $\prod_{i=1}^{k} \mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G])$. Let $f \in \mathrm{Hom}_{pwi}(G/Z(G), G')$. Denote by f_i, the homomorphism induced by f in $\langle\overline{x_i}\rangle$, for all $1 \le i \le k$. Since G is a nilpotent group of class 2, we have $[a^m, b] = [a, b]^m = [a, b^m]$ for each $a, b \in G$ and $m \in \mathbb{Z}$. Consequently, $[x_i^m, G] \le [x_i, G]$ for all $m \in \mathbb{Z}$ and $1 \le i \le k$. Therefore $f_i \in \mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G])$. Thus the map α sending any $f \in \mathrm{Hom}_{pwi}(G/Z(G), G')$ to $\alpha(f) = (f_1, \ldots, f_k) \in \prod_{i=1}^{k} \mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G])$ is well defined. Now we prove that this map is a monomorphism. Since $(fg)_i = f_i g_i$ for each $f, g \in \mathrm{Hom}_{pwi}(G/Z(G), G')$ and $1 \le i \le k$, α is homomorphism. Clearly, $\ker\alpha$ is trivial, this implies that α is monomorphism. Hence the proof of (i) is complete.

(ii) First we show that $[x_i, G]$ is finite if and only if $\langle\overline{x_i}\rangle$ is finite, and further

$\exp([x_i, G]) = \exp(\langle\overline{x_i}\rangle) = |\overline{x_i}|$. For this, let $|[x_i, G]| = n$. Since G is a nilpotent group of class 2, we have $[x_i^n, g] = [x_i, g]^n = 1$ for all $g \in G$ and so $x_i^n \in Z(G)$. Hence $\langle\overline{x_i}\rangle$ is finite and $|\overline{x_i}||n$. Conversely if $|\overline{x_i}| = m$ then $x_i^m \in Z(G)$ and $[x_i, G]^m = [x_i^m, G] = 1$. Consequently $[x_i, G]$ is finite and $\exp([x_i, G]) = n|m$. Therefore in this case, $m = n$. Hence by Lemma 2.1, for all $1 \leq i \leq k$ we have $\mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G]) \simeq \langle\overline{x_i}\rangle$ if and only if $[x_i, G]$ is cyclic.

Now from (i), we have a monomorphism from $\mathrm{Aut_{pwi}}(G)$ into $\prod_{i=1}^{k} \mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G])$ and therefore we conclude that there exists a monomorphism $\mathrm{Aut_{pwi}}(G) \hookrightarrow G/Z(G)$, this completes the proof of (ii).

If G is a d-group, then it is easy to see that the monomorphism defined in (i) is an isomorphism from $\mathrm{Aut_{pwi}}(G)$ into $\prod_{i=1}^{k} \mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G])$.

Finally to complete the proof, it is sufficient to show that if $\mathrm{Aut_{pwi}}(G) \simeq \mathrm{Inn}(G)$, then $[x_i, G]$ is cyclic for all $1 \leq i \leq k$. Since $\mathrm{Aut_{pwi}}(G) \simeq \mathrm{Inn}(G)$, by Proposition 1.2 we have $G/Z(G) \simeq \mathrm{Hom_{pwi}}(G/Z(G), G')$. On the other hand, G is a d-group and hence

$$\mathrm{Hom_{pwi}}(G/Z(G), G') \simeq \prod_{i=1}^{k} \mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G]).$$

It follows that

$$G/Z(G) = \langle\overline{x_1}\rangle \times \ldots \times \langle\overline{x_k}\rangle \simeq \prod_{i=1}^{k}\mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G]).$$

Now we may assume that $\langle\overline{x_1}\rangle \times \ldots \times \langle\overline{x_n}\rangle$ is the torsion part and $\langle\overline{x_{n+1}}\rangle \times \ldots \times \langle\overline{x_k}\rangle$ is the torsion-free part of $G/Z(G)$. Since for all $1 \leq i \leq n$, $\exp([x_i, G]) = \exp(\overline{x_i}) = |\overline{x_i}|$ and $\prod_{i=1}^{n} \mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G]) \simeq \langle\overline{x_1}\rangle \times \ldots \times \langle\overline{x_n}\rangle$, $\mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G]) \simeq \langle\overline{x_i}\rangle$ for all $1 \leq i \leq n$ and hence $[x_i, G]$ is cyclic. Furthermore, we have

$$\prod_{i=n+1}^{k} \mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G]) \simeq \langle\overline{x_{n+1}}\rangle \times \ldots \times \langle\overline{x_k}\rangle \simeq \mathbb{Z}^{\,k-n}.$$

Now we have $\mathrm{Hom}(\langle\overline{x_i}\rangle, [x_i, G]) \simeq [x_i, G]$, since $\langle\overline{x_i}\rangle \simeq \mathbb{Z}$ and hence $\prod_{i=n+1}^{m} [x_i, G] \simeq \mathbb{Z}^{\,k-n}$. That is $[x_i, G] \simeq \mathbb{Z}$ for all $n + 1 \leq i \leq k$. This implies that $[x_i, G]$ is cyclic for all $1 \leq i \leq k$, as required.

*Notice that if G is a finite group then, as a consequence of this result, we derive Theorem 3.5 and Corollary 3.6 of Yadav in [12].

The following corollary is an easy consequence of the above theorem.

Corollary 2.2 Let G be a finitely generated nilpotent group of class 2 with cyclic commutator subgroup. Then there exists a monomorphism from $\mathrm{Aut}_{\mathrm{pwi}}(G)$ into $\mathrm{Inn}(G)$ or equivalently $\mathrm{Aut}_{\mathrm{pwi}}(G)$ is isomorphic to a subgroup of $G/Z(G)$.

Remark 2.3 We keep here the notation used in Theorem 1.

(i) By the discussion of (ii) in Theorem 1, if G' is finite cyclic, then $G/Z(G)$ is finite and $|\mathrm{Aut}_{\mathrm{pwi}}(G)| \leq |\mathrm{Inn}(G)| = |G/Z(G)|$. On the other hand, $\mathrm{Inn}(G) \leq \mathrm{Aut}_{\mathrm{pwi}}(G)$ conclude that $\mathrm{Aut}_{\mathrm{pwi}}(G) = \mathrm{Inn}(G)$. Note that in this case, G is not necessarily finite.

(ii) If G' is infinite cyclic, it follows from the discussion of (ii) in Theorem 1, that $G/Z(G)$ is a free abelian group of finite rank, say $r(G/Z(G)) = k$. We certainly have $\mathrm{Inn}(G) \leq \mathrm{Aut}_{\mathrm{pwi}}(G)$ and thus $r(\mathrm{Inn}(G)) \leq r(\mathrm{Aut}_{\mathrm{pwi}}(G))$. Also $r(\mathrm{Aut}_{\mathrm{pwi}}(G)) \leq r(\mathrm{Inn}(G))$, since $\mathrm{Aut}_{\mathrm{pwi}}(G)$ is isomorphic to a subgroup of $\mathrm{Inn}(G)$. Therefore $\mathrm{Aut}_{\mathrm{pwi}}(G)$ and $\mathrm{Inn}(G)$ have the same rank and hence $\mathrm{Aut}_{\mathrm{pwi}}(G) \simeq \mathrm{Inn}(G)$.

Now it is easy to deduce Corollary 1 from Remark 2.3.

Remark 2.4 It is known that in a nilpotent groups of class 2, $\mathrm{Inn}(G) \leq \mathrm{Aut}_{\mathrm{pwi}}(G) \leq C^*$.So $\mathrm{Inn}(G) = \mathrm{Aut}_{\mathrm{pwi}}(G)$ when $\mathrm{Inn}(G) = C^*$. In [1] we characterized all non torsion-free finitely generated groups in which $\mathrm{Inn}(G) = C^*$. We proved that $\mathrm{Inn}(G) = C^*$ if and only if G is an abelian group or nilpotent of class 2 and $Z(G) \simeq C_m \times H \times \mathbb{Z}^r$ in which $C_m \simeq \Pi_{p \in \pi(G/Z(G))} Z(G)_p$, $H \simeq \Pi_{p \notin \pi(G/Z(G))} Z(G)_p$ and $r \geq 0$ is the torsion-free rank of $Z(G)$ and $G/Z(G)$ has finite exponent.

Hence if G is nilpotent group of class 2, $Z(G) \simeq C_m \times H \times \mathbb{Z}^r$ and $G/Z(G)$ has finite exponent then we have $\mathrm{Inn}(G) = \mathrm{Aut}_{\mathrm{pwi}}(G)$. Notice that in this case, G' is cyclic and the equality $\mathrm{Inn}(G) = \mathrm{Aut}_{\mathrm{pwi}}(G)$ also follows from Corollary 1.

Recall that by Corollary 3.6 in [12], in a finite nilpotent group of class 2, if G' is cyclic then $\mathrm{Aut}_{\mathrm{pwi}}(G) = \mathrm{Inn}(G)$. But we cannot hope for a similar conclusion when G is not finite.

For example, consider countably infinitely many copies $H_1, H_2, \ldots$ of a given nilpotent group H of class 2 with cyclic commutator subgroup. Let G (respectively, $\overline{G}$) be the direct product (the cartesian product) of the family $(H_i)_{i>0}$. Clearly, G and $\overline{G}$ are nilpotent of class 2. For each integer $i > 0$, choose an element $a_i \in H_i$ which is not in the center of H_i. Then the inner automorphism of $\overline{G}$ defined by $\overline{\alpha}((t_i)_{i>0}) = (a_i^{-1} t_i a_i)_{i>0}$ induces in G a pointwise inner automorphism α which is not inner (see [4]).

However, in a finitely generated nilpotent group of class 2 with cyclic commutator subgroup, we have $Aut_{pwi}(G) \simeq Inn(G)$, by Corollary 1. So the structure of $Aut_{pwi}(G)$ is determined.

Furthermore it is fairly easy to deduce Corollary 2 from Remark 2.3.

We end this part of the paper with some examples of infinite groups G satisfying the conditions of Corollary 1 and therefore $Aut_{pwi}(G) \simeq Inn(G)$.

Example 2.5 Let $G = \langle x_1, x_2, y_1, y_2 : x_1^p = x_2^p = y_1^p = 1, [x_1, x_2] = y_1, [y_1, y_2] = [x_i, y_j] = 1; 1 \leq i, j \leq 2 \rangle$. Then G satisfies the condition of Corollary 1. We have $G' = \langle y_1 \rangle \simeq C_p$, $Z(G) = \langle y_1, y_2 \rangle \simeq C_p \times \mathbb{Z}$ and $G/Z(G) = \langle \overline{x_1}, \overline{x_2} \rangle \simeq C_p \times C_p$ and hence $Aut_{pwi}(G) = Inn(G)$.

Example 2.6 Let $G = \langle x_1, x_2, x : [x_1, x_2] = x, [x_i, x] = 1; 1 \leq i \leq 2 \rangle$. Then G satisfies the condition of Corollary 1. We have $G' = Z(G) = \langle x \rangle \simeq \mathbb{Z}$ and $\frac{G}{Z(G)} = \langle \overline{x_1}, \overline{x_2} \rangle \simeq \mathbb{Z} \times \mathbb{Z}$ .Hence $Aut_{pwi}(G) \simeq Inn(G)$. It is easy to see that in this case every pointwise inner automorphism is inner and so $Aut_{pwi}(G) = Inn(G)$ (see [1, Example 3.4]).

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