

# A Complete Linear Connection Induced by Berwald Connection

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## Abstract

By using Berwald connection, we show that there are linear connections  $\nabla$  which are projectively equivalent and belong to the same projective structure on  $TM$ . We found a condition for the geodesics of the Berwald connection under which  $\nabla$  is complete.

## Introduction

Two (torsion free) linear connections  $D$  and  $\bar{D}$  on a smooth manifold  $M$  are said to be projectively equivalent if there exist a 1-form  $\rho$  on  $M$  such that

$$\bar{D} = D + \rho \otimes id + id \otimes \rho,$$

where  $id$  denotes the identity (1,1)-tensor on  $M$ . Projective equivalence is an equivalence relation on the set of torsion-free linear connections on  $M$ , and an equivalence class will be called a projective equivalence class [6]. Projective equivalence can be related to the concept of a projective structure. If  $M$  has dimension  $n$ , then a projective structure on  $M$  is a principal subbundle of the bundle of 2-frames over  $M$  having as its structure group the isotropy subgroup of  $PGL(n, R)$  at the origin of real projective space  $RP^n$ , [4].

According to this remark, we can introduce a projective structure on  $TM$ . Since the two (torsion free) linear connections on  $TM$  belong to the same projective structure on  $TM$  if and only if they are projectively equivalent, a projective equivalence class consists of those (torsion free) linear connections on  $TM$  which belong to the same projective structure on  $TM$ .

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In Finsler geometry, examples of important connections are proposed by L. Berwald [2], E. Cartan (1934), S. S. Chern [1] and Z. Shen [7]. Some of these connections are torsion free, for a list of almost all Finsler connections, one can refer to Bidabad and Tayebi [3]. So if we use a Finsler connection then we can show that there are many linear connections on  $TM$  contained in the same projective equivalence class on  $TM$  induced by this Finsler connection. For example in case of Berwald connection, we show that there is a linear connection  $\nabla$  on  $TM$  which is projectively equivalent to the Berwald connection and belong to the same projective structure on  $TM$ . We find a condition for the geodesics of the Berwald connection under which  $\nabla$  is complete ( to see a similar problem in the Riemannian case, refer to Spivak [6]).

### Preliminaries

Let  $M$  be an  $n$ -dimensional  $C^\infty$  manifold. Denote by  $T_x M$  the tangent space at  $x \in M$ , and by  $TM := \cup_{x \in M} T_x M$  the tangent bundle of  $M$ . Each element of  $TM$  has the form  $(x, y)$ , where  $x \in M$  and  $y \in T_x M$ . Let  $TM_0 = TM - \{0\}$ . The natural projection  $\pi: TM \rightarrow M$  is given by  $\pi(x, y) = x$ .

A (globally defined) Finsler structure [1] on a manifold  $M$  is a function  $F: TM \rightarrow [0, \infty)$

with the following properties:

- (i) **Regularity:**  $F$  is  $C^\infty$  on the entire slit tangent bundle  $TM_0$ .
- (ii) **Positive homogeneity:**  $F(x, \lambda y) = \lambda F(x, y)$  for all  $\lambda > 0$ .
- (iii) **Strong convexity:** The  $n \times n$  Hessian matrix

$$(g_{ij}) := \left( \left[ \frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive-definite at every point of  $TM_0$ .

Given a manifold  $M$  and a Finsler structure  $F$  on  $M$ , the pair  $(M, F)$  is called a Finsler manifold.  $F$  is called Riemannian if  $g_{ij}(x, y)$  are independent of  $y \neq 0$ .

Let  $M$  be a real  $n$ -dimensional connected manifold of  $C^\infty$ -class and  $(TM, \pi, M)$  its tangent bundle with zero section removed. Every local chart  $(U, \varphi = (x^i))$  on  $M$

induces a local chart  $(\varphi^{-1}(U), \varphi = (x^i, y^i))$  on  $TM$ . The kernel of linear map  $\pi_*: TTM \rightarrow TM$  is called the vertical distribution and is denoted by  $VTM$ . For every  $u \in TM$ ,  $\text{Ker } \pi_{*,u} = V_u TM$  is spanned by  $\{\frac{\partial}{\partial y^i}|_u\}$ . By a nonlinear connection on  $TM$  we mean a regular  $n$ -dimensional distribution  $H: u \in TM \rightarrow H_u TM$  which is supplementary to the vertical distribution i.e.

$$T_u(TM) = H_u TM \oplus V_u TM, \quad \forall u \in TM.$$

A basis for  $T_u TM$  adapted to the above direct sum is  $(\frac{\delta}{\delta x^i}|_u, \frac{\partial}{\partial y^i}|_u)$ , where

$$\frac{\delta}{\delta x^i}|_u = \frac{\partial}{\partial x^i} - N_j^i(u) \frac{\partial}{\partial y^j}|_u$$

and  $N_j^i$  are coefficients of the nonlinear connection. The dual basis of  $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$  is given by  $(dx^i, dy^i + N_j^i dx^j)$ . These are the Berwald bases.

### A complete linear connection

Let  $\nabla$  be a linear connection on a manifold  $M$ . A curve  $c: (a, b) \rightarrow M$  is an inextendible geodesic of  $\nabla$  iff  $c$  is a geodesic of  $\nabla$  and has no extension to  $[0, b + \alpha)$  as a geodesic of  $\nabla$  for any  $\alpha > 0$ . The connection  $\nabla$  is complete iff every geodesic of  $\nabla$  defined on a subinterval of  $R$  extends to a geodesic of  $\nabla$  defined on all of  $R$ .

In what follows, by using Berwald connection, we want to construct a linear connection  $\nabla$  on  $TM$  which are projectively equivalent and belong to the same projective structure on  $TM$ . We first define notion of Berwald connection.

Let  $M$  be a real  $n$ -dimensional  $C^\infty$  manifold and  $VTM = \bigcup_{v \in TM} V_v TM$  its vertical vector bundle. Suppose that  $HTM = \bigcup_{v \in TM} H_v TM$  is a non-linear connection on  $TM$ . The Berwald connection induced by a nonlinear connection with local coefficients  $N_j^i$  is a linear connection with the local coefficients  $\frac{\partial N_l^k}{\partial y^j}$ , (see [5]).

For example, consider  $S$  as a semispray with local coefficients  $G^i$  and  $N$  the induced nonlinear connection with local coefficients  $N_j^i = \frac{\partial G^i}{\partial y^j}$ . Since the nonlinear connection is symmetric then the Berwald connection  $D$  induced by  $N$  is a linear connection and has the expression

$$D \frac{\delta}{\delta x^i} \frac{\delta}{\delta x^j} = \frac{\partial^2 G^k}{\partial y^i \partial y^j} \frac{\delta}{\delta x^k}, \quad D \frac{\partial}{\partial y^i} \frac{\delta}{\delta x^j} = 0,$$

$$D \frac{\delta}{\delta x^i} \frac{\partial}{\partial y^j} = \frac{\partial^2 G^k}{\partial y^i \partial y^j} \frac{\partial}{\partial y^k}, \quad D \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} = 0.$$

**Theorem 3.1** Let  $D$  be a Berwald connection induced by the nonlinear connection  $N$  associated to a semispray and  $F$  a nonzero Finsler metric. Then there is a linear connection  $\nabla$  on  $TM$  defined by

$$\nabla_X Y = D_X Y + \frac{1}{2F} dF(X)Y + \frac{1}{2F} dF(Y)X, \quad \forall X, Y \in \chi(TM). \quad (1)$$

**Proof.** With respect to the Berwald basis,  $\nabla$  has the expression

$$\begin{aligned} \nabla \frac{\delta}{\delta x^i} \frac{\delta}{\delta x^j} &= \frac{\partial N_i^k}{\partial y^j} \frac{\delta}{\delta x^k} + \frac{1}{2F} \left( \frac{\delta F}{\delta x^i} \frac{\delta}{\delta x^j} + \frac{\delta F}{\delta x^j} \frac{\delta}{\delta x^i} \right) \\ \nabla \frac{\delta}{\delta x^i} \frac{\partial}{\partial y^j} &= \frac{\partial N_i^k}{\partial y^j} \frac{\partial}{\partial y^k} + \frac{1}{2F} \left( \frac{\delta F}{\delta x^i} \frac{\partial}{\partial y^j} + \frac{\partial F}{\partial y^j} \frac{\delta}{\delta x^i} \right) \\ \nabla \frac{\partial}{\partial y^i} \frac{\delta}{\delta x^j} &= \frac{1}{2F} \left( \frac{\partial F}{\partial y^i} \frac{\delta}{\delta x^j} + \frac{\delta F}{\delta x^j} \frac{\partial}{\partial y^i} \right) \\ \nabla \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j} &= \frac{1}{2F} \left( \frac{\partial F}{\partial y^i} \frac{\partial}{\partial y^j} + \frac{\partial F}{\partial y^j} \frac{\partial}{\partial y^i} \right) \end{aligned}$$

It is not difficult to show that the coefficients of  $\nabla$  satisfy the transformation law for the coefficients of a linear connection on  $TM$ .

For the linear connection (1), we consider the torsion  $T$ , defined as usual

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad \forall X, Y \in \chi(TM).$$

With respect to the Berwald basis we have

$$\begin{aligned} T\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) &= \left(\frac{\delta N_i^k}{\delta x^j} - \frac{\delta N_j^k}{\delta x^i}\right) \frac{\partial}{\partial y^k}, \\ T\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) &= 0, \\ T\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) &= 0, \\ T\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) &= 0. \end{aligned}$$

**Theorem 3.2** Let  $D$  be the Berwald connection and let  $\nabla$  be the linear connection defined in theorem 3.1. If every inextendible geodesic of  $D$ , such as  $c: [0, b) \rightarrow TM$ , has an orientation preservation reparametrization  $\sigma: [0, \infty) \rightarrow [0, b)$  such that  $c \circ \sigma$  is a geodesic of  $\nabla$ , then  $\nabla$  is complete.

**Proof.** Let  $c: [0, b) \rightarrow TM$  be the inextendible geodesic of  $D$  with  $c'(0) = (u, v)$ . There is an orientation preserving reparametrization  $\sigma: [0, \infty) \rightarrow [0, b)$  such that  $\tilde{c} = c \circ \sigma$  is a geodesic of  $\nabla$ . So we have  $\tilde{c}'(0) = \lambda c'(0)$  for some positive constant  $\lambda$ . Then  $\hat{c}: [0, \infty) \rightarrow TM$  given by  $\hat{c}(t) = \tilde{c}(\frac{t}{\lambda})$  is also a geodesic of  $\nabla$  and  $\hat{c}'(0) = (u, v)$ .

Thus  $\nabla$  is complete.

Let the hypotheses of 3.2 hold. If  $c: (a, b) \rightarrow TM$  is a geodesic of  $D$  and  $\sigma: (\alpha, \beta) \rightarrow (a, b)$  an orientation preserving reparametrization of  $c$  such that  $\tilde{c} = c \circ \sigma$  is a geodesic of  $\nabla$ , then  $\bar{\nabla}_{c'(t)} c'(t) = 0$ . Let  $t = \sigma(s)$  for  $s \in (\alpha, \beta)$ . So  $\tilde{c}'(s) = \sigma'(s)c'(\sigma(s)) = \frac{dt}{ds} \frac{dc}{dt} \Big|_{t=\sigma(s)}$ . Since

$$\nabla_{\frac{dc}{ds}} \frac{dc}{ds} = D_{\frac{dc}{ds}} \frac{dc}{ds} + \frac{1}{F} dF \left( \frac{dc}{ds} \right) \frac{dc}{ds}$$

thus

$$\frac{dt}{ds} \left( \left( \frac{dt}{ds} \right)^{-1} \frac{d^2 t}{ds^2} + \frac{d(\ln(F))}{ds} \right) \frac{dc}{dt} = \frac{dt}{ds} \left( \frac{d}{ds} \ln \left( F \frac{dt}{ds} \right) \right) \frac{dc}{dt} = 0.$$

This shows that  $F \frac{dt}{ds}$  is constant. As Finsler metric is positive function, so there is a constant  $C_1 > 0$  such that  $F \frac{dt}{ds} = \frac{1}{C_1}$ . This differential equation can be integrated to give

$$s(t) = \sigma^{-1}(t) = C_0 + C_1 \int_{t_0}^t F(c(\theta)) d\theta \quad (2)$$

where  $t_0 \in (a, b)$ ,  $C_0 \in \mathbb{R}$ .

**Theorem 3.3** Let  $F(x, y)$  be a nonzero Finsler metric, If for each inextendible geodesic of  $D$ , such as  $c: [0, b) \rightarrow TM$ , we have

$$\int_0^b F(c(t)) dt = \infty \quad (3)$$

then the connection  $\nabla$  defined by (1) is complete.

**Proof.** Suppose that the condition (3) holds. Let  $c: [0, b) \rightarrow TM$  be such a curve and let  $\sigma: [0, \beta) \rightarrow [0, b)$  be an orientation preserving reparametrization of  $c$  such that  $\tilde{c} = c \circ \sigma$  is a geodesic of  $\nabla$ . From (2),  $\sigma^{-1}(t)$  is given by

$$\sigma^{-1}(t) = C_1 \int_{t_0}^t F(c(\theta)) d\theta$$

with  $C_1 > 0$ . But  $\beta = C_1 \int_{t_0}^b F(c(\theta)) d\theta = \infty$ , so inextendible D-geodesic  $c$  has an orientation preserving reparametrization  $\sigma: [0, \infty) \rightarrow [0, b)$  such that  $c \circ \sigma$  is a geodesic of  $\nabla$  and so  $\nabla$  is complete.

We showed that two linear connections introduced in this paper, the Berwald connection  $D$  and the linear connection  $\nabla$  defined in theorem 3.1, are projectively equivalent and belong to the same projective structure on  $TM$ . We have also proved that for each inextendible geodesic of the Berwald connection such that the condition (3) holds then the connection  $\nabla$  is complete.

For example, let  $M = \mathbb{R}^n - \{0\}$  and let  $F$  be a nonzero Finsler metric such that for each  $x \in M$ ,  $f|_{T_x M}$  is a Minkowski norm on  $T_x M$ . Consider the curve  $c: [0, b) \rightarrow TM$  given by  $c(t) = p + t(x, y)$  where  $y \neq 0$  and  $b = \infty$ . With respect to this curve, it can be easily shown that the equation (3) is established.

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