

## Contractibility and idempotents in Banach algebras

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### Abstract

Let  $A$  be a Banach algebra. It is shown that a contractible ideal of a Banach algebra is complemented by its annihilator. Then, it is proved the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. Moreover, the notion of b-contractibility and one of its equivalent forms are introduced. Through an example, it is shown that b-contractibility is strictly weaker than contractibility.

### Introduction

Taylor in [13, Theorem 5.11] showed that a contractible Banach algebra with bounded approximation property is finite dimensional. Johnson in [6, Proposition 8.1] showed that a contractible commutative semisimple Banach algebra is finite dimensional. Curtis and Loy [1, Theorem 6.2] extended this result by dropping the semisimplicity assumption. But the question for noncommutative case has remained open. For more results of this type see [4], [5], [8], [10], [13].

This paper is organized as follows. In the second section, we show that a contractible ideal of a Banach algebra is controlled by its commutant and annihilator. Then, we prove the existence of minimal central idempotents in a contractible Banach algebra with a nonzero character. In the third section, we introduce a weaker version of contractibility which we call b-contractibility. We give a characterization of b-contractibility analog to that of contractibility given by Taylor. Also, we show that b-contractibility is strictly weaker than contractibility.

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First we recall some terminology. Throughout this paper,  $A$  is a Banach algebra and  $A$ -module means Banach  $A$ -bimodule. For a subset  $E$  of  $A$ ,  $E'$  is the commutant of  $E$ . If for every  $A$ -bimodule  $X$  every bounded derivation from  $A$  into  $X$  is inner, then  $A$  is called *contractible*. Also, the term "semisimple" means *Jacobson semisimple*. An idempotent  $e \in A$  is called *minimal* if  $eAe$  is a division ring. If  $e$  and  $f$  are idempotents in  $A$ , we write  $e \leq f$  if  $fe = ef = e$  holds. A nonzero idempotent  $e \in A$  is called *primitive* if  $0 \leq f \leq e$  implies that  $f = 0$  or  $f = e$ . Also, two idempotents  $e$  and  $f$  are said to be *orthogonal* if they satisfy  $ef = fe = 0$ . Let  $S$  be a subset of  $A$ . The *right annihilator* of  $S$  in  $A$  which we denote by  $\text{ran}(S)$  is the set

$$\text{ran}(S) = \{a \in A : ba = 0 \text{ for } b \in S\}.$$

The left annihilator  $\text{lan}(S)$  is defined similarly. The *annihilator* of  $S$  is the set  $\text{Ann}(S) = \text{ran}(S) \cap \text{lan}(S)$ .

## Contractibility

**Theorem 2.1.** Let  $A$  be a contractible Banach algebra which is an ideal in a Banach algebra  $B$ . Then  $A + A' = B$ .

*Proof.* If  $A + A' \neq B$ , then we can choose  $b \in B - (A + A')$ . Now define

$$D : A \rightarrow A, x \mapsto xb - bx.$$

Clearly  $D$  is a derivation on  $A$ . By assumption there exists an  $a \in A$  such that  $D(x) = xa - ax$  for all  $x \in A$ . The latter result implies that  $b - a \in A'$  or equivalently  $b \in A + A'$  which contradicts the selection of  $b$ . Therefore  $A + A' = B$ .

**Theorem 2.2.** Let  $A$  be a contractible Banach algebra which is an ideal in a Banach algebra  $B$ . Then  $B = A \oplus \text{Ann}(A)$ .

*Proof.* Since  $A$  is contractible then  $M_2(A)$  with  $l^1$ -norm is contractible, where  $M_2(A)$  is the algebra of  $2 \times 2$  matrices with the entries from  $A$ . On the other hand  $M_2(A)$  is an ideal in  $M_2(B)$  and by Theorem 2.1 we have the equality  $M_2(B) = M_2(A) + M_2(A)'$ . One can easily observe that

$$M_2(A)' = \begin{bmatrix} A' & Ann(A) \\ Ann(A) & A' \end{bmatrix}.$$

Thus  $B = A + Ann(A)$ . But  $A \cap Ann(A) = 0$ , because  $A$  is unital. Therefore the identity  $B = A \oplus Ann(A)$  holds.

**Remark.** In Theorems 2.1 and 2.2,  $A$  and  $B$  are related only algebraically. Indeed if there exists an infinite dimensional contractible Banach algebra  $A$  which is an ideal in a Banach algebra  $B$ , then the norm topology of  $A$  could be different from the relative norm topology of  $A$  which inherits from  $B$ .

**Theorem 2.3.** Let  $A$  be a contractible Banach algebra which admits a nonzero multiplicative linear functional  $f$ . Then  $A$  contains a central minimal idempotent.

*Proof.* Let  $d = \sum_{n=1}^{\infty} a_n \otimes b_n$  be a diagonal for  $A$  and define

$$T : A \rightarrow A \mapsto \sum_{n=1}^{\infty} \langle f, aa_n \rangle b_n.$$

Since  $\sum_n a_n b_n = 1$ , then

$$\begin{aligned} \langle f, T(1) \rangle &= \langle f, \sum_n \langle f, a_n \rangle b_n \rangle = \sum_n \langle f, a_n \rangle \langle f, b_n \rangle \\ &= \sum_n \langle f, a_n b_n \rangle = \langle f, \sum_n a_n b_n \rangle = \langle f, 1 \rangle = 1. \end{aligned}$$

Thus  $T(1) \neq 0$ . Moreover for every  $a \in A$  and  $g, h \in A^*$  we have

$$\begin{aligned} \langle h, \sum_n \langle g, aa_n \rangle b_n \rangle &= \sum_n \langle g, aa_n \rangle \langle h, b_n \rangle \\ &= \langle g \otimes h, \sum_n aa_n \otimes b_n \rangle \\ &= \langle g \otimes h, \sum_n a_n \otimes b_n a \rangle \\ &= \sum_n \langle g, a_n \rangle \langle h, b_n a \rangle \\ &= \langle h, \sum_n \langle g, a_n \rangle b_n a \rangle. \end{aligned}$$

This implies that

$$\sum_n \langle g, aa_n \rangle b_n = \sum_n \langle g, a_n \rangle b_n a.$$

Thus we assume that

$T(1) = e$ , then we have  $T(a) = \sum_n \langle f, aa_n \rangle b_n = \sum_n \langle f, a_n \rangle b_n a = ea$ . On the other hand we have  $T(a) = \sum_n \langle f, aa_n \rangle b_n = \langle f, a \rangle \sum_n \langle f, a_n \rangle b_n = \langle f, a \rangle e$ . Hence  $T$  is an operator of rank one and  $e^2 = T(e) = \langle f, e \rangle e = e$ . Now define

$$T_1 : A \rightarrow A, a \mapsto \sum_n a_n \langle f, aa_n \rangle.$$

With a similar argument we can show that

$$T_1(a) = ae' = \langle f, a \rangle e' \quad a \in A$$

where  $e' = T_1(1)$ . Also we have  $e'^2 = e'$  and  $\langle f, e' \rangle = 1$ . Now the identities

$$ee' = \langle f, e' \rangle e = e, \quad ee' = \langle f, e \rangle e' = e'$$

imply that  $e = e'$  and for every  $a \in A$  we have

$$ea = \langle f, a \rangle e = \langle f, a \rangle e' = ae' = ae.$$

Therefore  $e$  is a central idempotent. In addition since  $T$  is a rank one operator and  $\text{ran } T = eAe$ , then  $eA = eAe = Ce$  is a division ring. Therefore  $e$  is a minimal idempotent.

### b-Contractibility

**Definition.** Let  $A$  be a Banach algebra and  $\pi$  be the natural map,

$$\pi : A \otimes A \longrightarrow A, \quad \pi \left( \sum_n a_n \otimes b_n \right) \rightarrow \sum_n a_n b_n.$$

Let  $b \in A$  and  $X$  be an  $A$ -module. We say that a derivation  $\delta : A \longrightarrow X$  is a *b-derivation* if there exists another derivation  $\delta' : A \longrightarrow X$  such that  $\delta = b\delta'$ , where  $(b\delta')(a) = b\delta'(a)$ . Also we say that  $A$  is *b-contractible* if for every  $A$ -module  $X$ , every bounded  $b$ -derivation from  $A$  into  $X$  is inner. We call  $d \in \hat{A} \otimes A$  a *b-diagonal* if  $\pi(d) = b$  and  $a.d = d.a$  for all  $a \in A$ .

**Theorem 3.1.** Let  $A$  be a unital Banach algebra and  $b \in A' - \{0\}$ . Then  $A$  is *b-contractible* if and only if  $A$  has a *b-diagonal*.

*Proof.* First suppose  $A$  is *b-contractible* and  $\pi$  is defined as above. Clearly  $\ker \pi$  is an  $A$ -module and if we define

$$\delta : A \rightarrow \ker \pi, a \mapsto ab \otimes 1 - b \otimes a$$

then it is easy to see that  $\delta$  is a *b-derivation*. Indeed  $\delta = b\delta'$  where

$$\delta' : A \rightarrow \ker \pi, a \mapsto a \otimes 1 - 1 \otimes a$$

since  $A$  is *b-contractible*, then there exists an element  $\sum_n c_n \otimes d_n \in \ker \pi$  such that

$$\delta(a) = \sum_n ac_n \otimes d_n - \sum_n c_n \otimes d_na \quad a \in A.$$

Let  $d = b \otimes 1 - \sum_n c_n \otimes d_n$ . The above identities show that  $\pi(d) = b$  and  $a.d = d.a$  for all  $a \in A$ . Therefore,  $d$  is a  $b$ -diagonal for  $A$ . Conversely suppose  $d = \sum_n a_n \otimes b_n$  is a  $b$ -diagonal for  $A$ ,  $X$  is an  $A$ -module and  $\delta : A \longrightarrow X$  is a bounded derivation. Clearly the map

$$\psi : A \times A \rightarrow X, (a, c) \mapsto a\delta(c)$$

is a bounded bilinear map. So by the universal property of projective tensor product there is a linear map  $\varphi : A \hat{\otimes} A \longrightarrow X$  such that  $\varphi \circ \otimes = \psi$  that is  $\varphi(a \otimes c) = a\delta(c)$ . In particular using the fact that  $d$  is a  $b$ -diagonal for  $A$ , we get

$$\sum_n aa_n\delta(b_n) = \varphi(a.d) = \varphi(d.a) = \sum_n a_n\delta(b_na), \quad a \in A.$$

Now if  $x = \sum_n a_n\delta(b_n)$ , then for every  $a \in A$  we have

$$ax - xa = \sum_n aa_n\delta(b_n) - \sum_n a_n\delta(b_n)a = \sum_n aa_n\delta(b_n) + b\delta(a) - \sum_n a_n\delta(b_na).$$

Thus the identity  $ax - xa = b\delta(a)$  holds for every  $a \in A$ . Therefore every  $b$ -derivation is inner.

**Example 3.2.** Let  $A$  be the Banach algebra  $l_1(N)$  with pointwise multiplication and  $\{e_n\}$  be the standard basis for  $A$ . Then for every positive integer  $n$ ,  $A$  is  $e_n$ -contractible. Indeed  $e_n \otimes e_n$  is an  $e_n$ -diagonal for  $A$ . But  $A$  is not contractible, since it is not unital. Therefore  $b$ -contractibility does not imply contractibility.

**Remark.** If  $A$  is contractible, then it is unital and one can easily observe that  $A$  is  $b$ -contractible for every  $b \in A - \{0\}$ . However the above example shows that for non-unital Banach algebras the converse is not true. We do not know whether this is true for unital Banach algebras or not.

**Problem.** Does there exist a unital Banach algebra which is  $b$ -contractible for some nonzero central idempotent  $b$ , but is not contractible?

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