

Hypergroup Structures with Regular Multiplications

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Abstract

In Banach algebras, the group algebra $L(G)$ is Arens regular if and only if G is finite. In this paper, the researcher has obtained a hypergroup structure (in the sense of Dunkl) whose measure algebra has regular multiplication. The most interesting result was that if $L(X)$ is Arens regular then the convolution is Arens regular as a bilinear map. The condition obtained gives regularity of multiplication in the Hypergroup, which X is not finite.

Introduction

The regularity of a bounded bilinear mapping was defined by Arens (see [1]). For some important Banach algebras, the first and the second Arens product, on their second dual, are different. Therefore, these algebras are not Arens regular. A number of Banach algebras, commonly occurring in functional and harmonic analysis, are not Arens regular. The group algebra $L(G)$ of a locally compact Hausdorff group is Arens regular if and only if G is finite (see [4], [13], [14]). In [9], the researcher has shown that the hypergroup algebra $L(X)$, where X is a locally compact Hausdorff space, can be Arens regular without X being finite. Also, in [10], for a general measure algebra \mathcal{L} , in $M(X)$, if $e \in X$ is not isolated in $\text{supp } \mathcal{L}$, and that δ_e acts as an identity for \mathcal{L} , then \mathcal{L} is not Arens regular. These are not the only ways to construct the regular or irregular multiplications. In [8], for the circle group T , two multiplications have been constructed on $M(T)$, one of which is regular and the other is irregular. In the present note, we obtain a hypergroup structure whose measure algebra has regular multiplication. Some related results can be

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found in [8],[9],[10]. Also, basic facts about measure algebras on hypergroups can be found in [6], [7], [11].

The researcher will begin with the Arens multiplications and the hypergroup structures.

1- Other Versions of Arens Multiplications

In [1], Arens showed how the multiplication of a Banach algebra could be extended to a multiplication on the second dual. His method was essentially algebraic, and this is indeed the easiest way to prove that the construction works. However, we shall describe the results. See [12] for details.

Let A be a set with a multiplication $(x, y) \rightarrow x.y$. Let B be a set with $A \subseteq B$, under its topology, A is dense in B . For x, y in B , take $(\tau_\alpha), (s_\beta)$ in A such that $\lim_\alpha \tau_\alpha = x, \lim_\beta s_\beta = y$. Then, the first extension of multiplication is given by

$$\tau_\alpha.y = \lim_\beta \tau_\alpha.s_\beta, \quad x.y = \lim_\alpha \lim_\beta \tau_\alpha.s_\beta,$$

while the second extension is given by

$$x \circ s_\beta = \lim_\alpha \tau_\alpha.s_\beta, \quad x \circ y = \lim_\alpha \tau_\alpha.s_\beta.$$

The set A is Arens regular (or, briefly, regular) if $x.y = x \circ y$. By the above definition for a and b in B , the product $a.b$ (resp. $a \circ b$) is continuous in the a (resp. b) variable for each fixed b (resp. a) in B . Generally, $a.b$ (resp. $a \circ b$) is not continuous in b (resp. a) when $a \notin A$ (resp. $b \notin A$). This suggests the first result about regularity, which is entirely elementary.

Proposition 1.1. Let $A \subseteq B$ and let A be dense in B . Then.

- (i) if A is commutative then A is Arens regular if and only if B is commutative.
- (ii) A is Arens regular if and only if the multiplication in B is continuous in each variable (without any restriction on the other).
- (iii) Let B be compact. Then, A is Arens regular if and only if the multiplication in B separately sequentially continuous.

Proof. See [5] and [12].

Regularity of Banach algebras 1.2. Let A be a normed space over K ($K=\mathbb{R}$ or $K=\mathbb{C}$). The dual space A^* ; i.e. A^* , is the vector space $\Lambda(A^*, K)$ equipped with the norm

$$\|f\| = \sup\{|f(x)| : x \in A, \|x\| \leq 1\}.$$

Thus, A^* is a Banach space. Let $(A^*)^*$ be the dual space of A^* , $(A^*)^* = \Lambda(A^*, K)$. Since A^* is itself a Banach space, it is susceptible to the same construct; i.e. one can form $(A^*)^* = A^{**}$; this is also a Banach space, called the dual or bidual of A , and denoted by A^{**} . This can go on.

For each $x \in A$, the value of an element $x^{**} \in A^{**}$ is defined by $x^{**}(f) = f(x)$ for all $f \in A^*$. So x^{**} is linear on A^* , and $\|x^{**}\| = \|x\|$. Thus, the canonical embedding mapping $x \rightarrow x^{**}$ preserves norms and an isometric from A into its second dual A^{**} . Therefore, we can regard A as a subspace of A^{**} .

Let $\sigma(A^{**}, A^*)$ be the weak*-topology on A^{**} . By [3], A is weak*-dense in A^{**} . So, for $F \in A^{**}$, $G \in A^{**}$, we can find two bounded nets $(\mu_\alpha), (v_\beta)$ in A with $F = \omega^* - \lim_\alpha \mu_\alpha, G = \omega^* - \lim_\beta v_\beta$. The topological extension of first and second Arens product are given by

$$FG = \omega^* - \lim_\alpha \omega^* - \lim_\beta \mu_\alpha v_\beta, \quad F \circ G = \omega^* - \lim_\beta \omega^* - \lim_\alpha \mu_\alpha v_\beta.$$

Thus, the order in which the limits are taken distinguishes between the extensions.

Moreover, the first Arens product is characterized by the two properties:

- (i) for each $G \in A^{**}$, the map $F \rightarrow FG$ is weak*-continuous on A^{**} .
- (ii) For each $\mu \in A$, the map $G \rightarrow \mu G$ is weak*-continuous on A^{**} .

The second Arens product is defined similarly. Therefore, the second dual A^{**} of A can be given the Banach algebra structure by means of the first (or second) Arens product.

Now, we want to describe Arens products as an algebraic extension. Indeed, for $F, G \in A^{**}$, $f \in A^*$, and $\mu, v \in A$, one can find $FG, F \circ G$ successively as follows:

$$\begin{aligned} \langle FG, f \rangle &= \langle F, Gf \rangle, \langle Gf, \mu \rangle = \langle G, f\mu \rangle, \langle f\mu, v \rangle = \langle f, \mu v \rangle, \\ \langle F \circ G, f \rangle &= \langle G, f \circ F \rangle, \langle f \circ F, \mu \rangle = \langle F, \mu \circ f \rangle, \langle \mu \circ f, v \rangle = \langle f, v\mu \rangle. \end{aligned}$$

So, a Banach algebra is said to have regular multiplication if $FG = F \circ G$.

A^{**} is not compact in the weak*-topology. But the closed unit ball of A^{**} is weak-compact [3]. So, by definition or [5], we have:

Proposition 1.3. Let A be commutative. A is Arens regular if and only if the first or second Arens product in A^{**} is weak*-continuous in each variable.

Proof. For $F, G \in A^{**}$, There are two nets (μ_α) and (ν_β) in A which weak*-converge to F and G . So, A is Arens regular if and only if $FG = F \circ G$. It is equivalent to this fact; for all $f \in A^*$,

$$\lim_{\alpha} \lim_{\beta} f(\mu_\alpha \nu_\beta) = \lim_{\beta} \lim_{\alpha} f(\mu_\alpha \nu_\beta).$$

2- Hypergroup Structures

Let X be a locally compact Hausdorff space and $M(X)$ denotes the set of all bounded, regular, complex Borel measures on X . For each μ and ν in $M(X)$, $\mu * \nu$ denotes the convolution of μ and ν . Let δ_r be the unit mass at r . The product formulas of the type.

$$\mu * \nu(f) = \int_x \int_x (\delta_x * \delta_y)(f) d\mu(x) d\nu(y)$$

On $M(X)$ becomes a Banach algebra. Dunkl (1972) and Jewett. (1975) have shown how one defines a product on $M(X)$, which makes it a Banach algebra. In some cases, an investigation begins with a convolution algebra of measures as the primitive object, upon which to build a theory; this is the case of the analysis of the objects called hypergroups which are generalizations of the convolution algebra of Borel measures on a group. One of the objects of this paper will be the introduction of a large class of new convolution structures, many of which are new hypergroups.

Let $C_b(X)$, $C_0(X)$ and $C_c(X)$ denote the spaces of continuous functions on X which are bounded, those which vanish at infinity and those having compact support respectively. By $M(X)$ and $M_p(X)$, we abbreviate the space of Radon measures and probability measures on X .

Definition 2.1. A hypergroup $(X, *)$ is a Banach algebra of the Borel measures $M(X)$ on a locally compact Hausdorff space X with product $*$ called convolution it satisfies the following axioms:

- (i) There is a map $\lambda: X \times X \rightarrow M_p(X)$ with for every $x, y \in X$, the measures $\lambda_{(x,y)}$ have compact supports and $\lambda_{(x,y)} = \lambda_{(y,x)}$;

(ii) for each $f \in C_c(X)$, the map $(x, y) \rightarrow \lambda_{(x,y)}(f)$ is in $C_b(X \times X)$ and $r \rightarrow \lambda_{(x,y)}(f)$ is in $C_c(X)$, for every $y \in X$;

(iii) the convolution $(\mu, \nu) \rightarrow \mu * \nu$ (or $\mu \nu$) of measures defined by

$$\mu * \nu(f) = \int_X \int_X \lambda_{(x,y)}(f) d\mu(x) d\nu(y), \quad (\mu, \nu \in M(X), f \in C_o(X))$$

is associative (and clearly $\lambda_{(x,y)} = \delta_x * \delta_y$)

(iv) there is a unique $e \in X$ such that $\lambda_{(x,y)} = \delta_x$ for all $x \in X$.

In [7], Ghahramani and Medghalchi have constructed and studied the subalgebra of $M(X)$ which is determined in the following way:

$$L(X) = \{ \mu \in M(X) : x \rightarrow |\mu| * \delta_x, x \rightarrow \delta_x * |\mu| \text{ are norm-continuous} \}.$$

This algebra generalizes the algebra $L(G)$ of $M(G)$ for the locally compact topological groups G . They have shown that $L(X)$ is a Banach subalgebra of $M(X)$ and it has a bounded approximate identity of norm 1. Therefore, $L(X)$ can be regarded as a subspace of $L(X)^{**}$ and then $L(X)$ is weak*-dense in $L(X)^{**}$ in [11], Medghalchi studied the second dual of $L(X)$.

Let A, B, C , be disjoint. And e, z be single points not in $A \cup B \cup C$. Write $X = \{e\} \cup A \cup B \cup C \cup \{z\}$. Let X be a compact Hausdorff space with e and z as isolated points. Each $\mu \in M(X)$ can be written in the unique form

$$\mu_e \delta_e + \mu_B + \mu_C + \mu_z \delta_z$$

where, μ_A, μ_B, μ_C are the restriction of μ to A, B, C , respectively, and μ_e and μ_z are scalars.

The following Theorem gives the structure of a hypergroup on the locally compact space X .

Theorem 2.2. Let $\lambda: A \times B \rightarrow (M_y(C), \text{weak}^*)$ be continuous and the map $x \rightarrow \lambda_{(x,y)}(f)$ is in $C_c(C)$ for every $x, y \in A \cup B, f \in C(C)$ and $\lambda(a,b) = \lambda(b,a)$ ($a \in A, b \in B$). There is a hypergroup structure on X such that $M(X - \{e\}) = L(X)$ if and only if $M(A \cup B) \subseteq L(X)$.

Proof. We define a multiplication on $M(X)$ by the following way:

take a map $\tilde{\lambda}: X \times X \rightarrow M_p(X)$ with two properties.

(i) $\tilde{\lambda}_{(x,e)} = \tilde{\lambda}_{(e,x)} = \delta_x$ ($x \in X$);

$$(ii) \tilde{\lambda}_{(x,y)} = \begin{cases} \lambda_{(x,y)} & (x \in A, y \in B) \\ \delta_Z & \text{otherwise} \end{cases}$$

it is clear that $\|\tilde{\lambda}_{(x,y)}\| = 1$ for all $x, y \in X$.

Then, we define a multiplication $\mu\nu$, for $\mu, \nu \in M(X)$, by

$$\mu\nu = \int_X \int_X \tilde{\lambda}_{(x,y)} d\mu(x) d\nu(y)$$

It is easy to show that the multiplications is commutative and is the identity. Now, we prove that it is associative.

If one of the x, y, z , (in X) is equal to e , then

$$(\delta_x * \delta_y) * \delta_z = \delta_x * (\delta_y * \delta_z)$$

Now, suppose that μ, ν, ξ are in $M(X - \{e\})$. Therefore,

$$\begin{aligned} \mu &= \mu_A + \mu_B + \mu_C + \mu_Z \delta_Z, \\ \nu &= \nu_A + \nu_B + \nu_C + \nu_Z \delta_Z, \\ \xi &= \xi_A + \xi_B + \xi_C + \xi_Z \delta_Z. \end{aligned}$$

Suppose $Y = A \cup C \cup \{z\}$. We have

$$\begin{aligned} \mu_A \nu_Y &= \int_X \int_X \tilde{\lambda}(x, y) d\mu_A(x) d\nu_Y(y) \\ &= \int_X \int_X \delta_z d\mu_A(x) d\nu_Y(y) \\ &= \mu_A(I) \nu_Y(I) \delta_Z. \end{aligned}$$

Similarly $\mu_Y \nu_A = \mu_Y(I) \nu_A(I) \delta_Z$ and if $T = B \cup C \cup \{z\}$ then

$$\mu_T \nu_B = \mu_T(I) \nu_B(I) \delta_Z, \mu_B \nu_T = \mu_B(I) \nu_T(I) \delta_Z.$$

Let $'*'$ be the convolution arising from the multiplication in A and B (or in B and A) i.e.,

$$\begin{aligned} \mu_A * \nu_B &= \mu_A \nu_B = \int_X \int_X \tilde{\lambda}(x, y) d\mu_A(y) d\nu_B(x), \\ \mu_A * \nu_B(f) &= \int_A \int_B \int_C f(t) d\lambda_{(x,y)}(t) d\nu_B(x) d\mu_A(y). \end{aligned}$$

Hence, $\text{supp}(\mu_A * \nu_B) \subseteq C$. Therefore, we have

$$\begin{aligned} \mu\nu &= (\mu_A + \mu_B + \mu_C + \mu_Z \delta_Z)(\nu_A + \nu_B + \nu_C + \nu_Z \delta_Z) \\ &= \mu_A \nu_Y + \mu_A * \nu_B + \mu_B \nu_T + \mu_B * \nu_A + \mu_Z \delta_Z \nu \\ &= \mu_A(I) \nu_Y(I) \delta_Z + \mu_B(I) \nu_T(I) \delta_Z + \mu_Z \delta_Z(I) \nu(I) \delta_Z + \mu_A * \nu_B + \mu_B * \nu_A \\ &= [\mu(I) \nu(I) - \mu_A(I) \nu_B(I) - \mu_B(I) \nu_A(I)] \delta_Z + \mu_A * \nu_B + \mu_B * \nu_A \end{aligned}$$

On the other hand,

$$\begin{aligned} \mu_A * \nu_B(1) &= \int_A \int_B \int_C d\lambda_{(x,y)}(t) d\nu_B(x) d\mu_A(y). \\ &= \int_A \int_B d\nu_B(x) d\mu_A(y) = \mu_A(1)\nu_B(1). \end{aligned}$$

Hence,

$$\begin{aligned} (\mu\nu) \psi &= [\mu(1)\nu(1) - \mu_A(1)\nu_B(1) - \mu_B(1)\nu_A(1)]\delta_z \psi + (\mu_A * \nu_A + \mu_B * \nu_B) \psi \\ &= \mu(1)\nu(1)\psi(1)\delta_z. \end{aligned}$$

Similarly $\mu(\nu\psi) = \mu(1)\nu(1)\psi(1)\delta_z$. So that, the multiplication is associative

Now, let $f \in C_c(X)$, then we have

$$\begin{aligned} \int_X f(t) d\tilde{\lambda}_{(x,e)}(t) &= f(x), \\ \int_X f(t) d\tilde{\lambda}_{(x,y)}(t) &= \begin{cases} d\lambda_{(x,y)}(f) & (x \in A, y \in B) \\ f(z) & \text{otherwise.} \end{cases} \end{aligned}$$

Then the map $(x, y) \rightarrow \tilde{\lambda}_{(x,e)}(f)$ is in $C_b(X \times X)$ and the map $x \rightarrow \tilde{\lambda}_{(x,y)}(f)$ is in $C_c(X)$ for all $y \in X$. Therefore, X is a hypergroup.

We now prove that $L(X) = M(X - \{e\})$. Since X is not discrete, $\delta_e \notin L(X)$ (Theorem 2, [9]). Let $\mu \in M(X - \{e\})$. Then

$$\delta_x |\mu| = \begin{cases} |\mu| & (x = e) \\ |\mu|(1)\delta_z & (x = z). \end{cases}$$

Now, if $x \in X - \{e, z\}$ then,

$$\delta_x |\mu| = \begin{cases} |\mu|(1) - |\mu_B|(1)\delta_z + \delta_x * |\mu_B| & (x = A) \\ |\mu|(1) - |\mu_A|(1)\delta_z + \delta_x * |\mu_A| & (x = B). \end{cases}$$

Hence, for $x=y=z$ or $x=y=e$,

$$\|\delta_x |\mu| - \delta_y |\mu|\| = 0$$

Otherwise,

$$\|\delta_x |\mu| - \delta_y |\mu|\| = \begin{cases} \|\delta_x |\mu_B| - \delta_y |\mu_B|\| & (x, y \in A) \\ \|\delta_x |\mu_A| - \delta_y |\mu_A|\| & (x, y \in B). \end{cases}$$

Therefore, $\mu \in L(X)$ if and only if $\mu_A \in L(X), \mu_B \in L(X)$. This statement is equivalent to, $\mu \in L(X)$ if and only if $M(A \cup B) \subseteq L(X)$. So, the conclusion holds.

Let $M(X)$ be the space of bounded regular Borel measures. We shall say that $M(X)$ has a general measure multiplication, if there exists a bilinear associative map

$\phi: M(X) \times M(X) \rightarrow M(X)$ such that

$$\phi(M_p(X) \times M_p(X)) \subseteq M_p(X).$$

Also, we shall say that ϕ is Arens regular, if for every two nets $(\mu_\alpha), (v_\beta)$ in $M(X)$,

$$w^* - \lim_{\alpha} w^* - \lim_{\beta} \phi(\mu_\alpha, v_\beta) = w^* - \lim_{\beta} w^* - \lim_{\alpha} \phi(\mu_\alpha, v_\beta),$$

when, both exist (see introduction [10]).

Theorem 2.3. Let $M(A \cup B) \subset L(X)$. Then, $L(X)$ is Arens regular if and only if there is a bilinear map $\phi: M(A) \times M(B) \rightarrow M(C)$ which is Arens regular.

Proof. By theorem 1, $L(X) = M(X - \{e\})$ and (by the lemma, 4, [9]),

$$L(X) = M(A) \oplus M(B) \oplus M(C) \oplus \square \delta_z,$$

$$L(X)^{**} = M(A)^{**} \oplus M(B)^{**} \oplus M(C)^{**} \oplus \square \delta_z.$$

So, each $\mu \in L(X), F \in L(X)^{**}$, can be written uniquely in the form

$$\mu = \mu_A + \mu_B + \mu_C + \mu_z \delta_z,$$

$$F = F_A + F_B + F_C + \mu_z \delta_z,$$

where, μ_A is the restriction of μ to A and F_A is the restriction of F to $M(A)^*$ and so on.

Let $\mu, v \in M(X)$. We define $\Phi: M(X) \times M(X) \rightarrow M(X)$ by

$$\Phi(\mu, v) = \mu v = \int_X \int_X \tilde{\lambda}(x, y) d\mu(x) dv^*(y).$$

If $\mu, v \in M_p(X)$ then $\phi(\mu, v) \in M_p(X)$. Thus, the multiplication Φ maps probability measures to probability measures.

First, suppose that $L(X)$ is Arens regular. So, for every nets $(\mu_\alpha) \subset M(A), (v_\beta) \subset M(B)$, if

$$w^* - \lim_{\alpha} w^* - \lim_{\beta} \Phi(\mu_\alpha, v_\beta), w^* - \lim_{\beta} w^* - \lim_{\alpha} \Phi(\mu_\alpha, v_\beta)$$

exist, and then they are equal (by Theorem 1, [5]).

Conversely, let Φ be Arens regular and $F, G \in L(X)^{**}$. There are two nets (μ_α) and (v_β) in $L(X)$ whit

$$w^* - \lim_{\alpha} \mu_\alpha = F, w^* - \lim_{\beta} \mu_\beta = G,$$

$$\text{supp } \mu_\alpha \subseteq X - \{e\}, \text{supp } \mu_\beta \subseteq X - \{e\}$$

The multiplication Φ is Arens regular. Therefore,

$$w^* - \lim_{\alpha} w^* - \lim_{\beta} \phi((\mu_A)_\alpha, (v_B)_\beta) = w^* - \lim_{\beta} w^* - \lim_{\alpha} \phi((\mu_A)_\alpha, (v_B)_\beta),$$

$$w^* - \lim_{\alpha} w^* - \lim_{\beta} \phi((v_B)_\alpha, (\mu_A)_\alpha) = w^* - \lim_{\beta} w^* - \lim_{\alpha} \phi((v_B)_\alpha, (\mu_A)_\alpha).$$

Combining the above equalities, we have

$$\begin{aligned} FG &= w^* - \lim_{\alpha} w^* - \lim_{\beta} \mu_{\alpha} \nu_{\beta} \\ &= w^* - \lim_{\alpha} w^* - \lim_{\beta} [(\mu_{\alpha}(1)\nu_{\beta}(1) - (\mu_A)_{\alpha}(1)(\nu_B)_{\beta}(1) - (\mu_B)_{\alpha}(1)(\nu_A)_{\alpha}(1))\delta_z + \\ &\quad \phi((\mu_A)_{\alpha}(\nu_B)_{\beta}) + \phi((\mu_B)_{\alpha}(\nu_A)_{\beta})] = w^* - \lim_{\beta} w^* - \lim_{\alpha} \mu_{\alpha} \nu_{\beta} = GF. \end{aligned}$$

Thus, $FG=GF$. By (Propositon1, [5]), $L(X)$ is Arens regular.

Now, let A and C be disjoint, $X=\{e\} \cup A \cup C \cup \{z\}$ and $e, z \notin A \cup C$. Whith the topology of X , A and C are compact subspaces of X and e, z are isolated points.

Theorem2.4. If $\lambda : A \times A \rightarrow M_p(X)$ is weak*-continuous and symmetric, then there exists a hypergroup structure on X so that

(i) $M(A) \subseteq L(X)$ if and only if $M(X-\{e\})=L(X)$;

(ii) Let $M(A) \subseteq L(X)$. Then there exists a bilinear associative map

$\Phi : M(A) \subseteq M(A) \rightarrow M(C)$ which maps probability measures to probability measures and $L(X)$ is Arens regular if and only if Φ is Arens regular.

Proof. Define $\tilde{\lambda} : X \times X \rightarrow M_p(X)$ with the following equations:

$$\begin{aligned} \text{(i)} \quad \tilde{\lambda}_{(x,e)} &= \tilde{\lambda}_{(e,x)} = \delta_x; \\ \text{(ii)} \quad \tilde{\lambda}_{(x,y)} &= \begin{cases} \lambda^{(x,y)} & (x, y \in A) \\ \delta_z & (\text{otherwise}). \end{cases} \end{aligned}$$

Then, we define a convolution $\mu\nu$ for $\mu, \nu \in M(X)$ by

$$\mu\nu = \int_X \int_X \tilde{\lambda}_{(x,y)} d\mu(x) d\nu(y).$$

It is clear that, if $\mu, \nu \in M(X-\{e\})$ and then

$$\mu\nu = (\mu(1)\nu(1) - \mu_A(1)\nu_A(1))\delta_z + \mu_A * \nu_A.$$

The rest of the proof is the same as the proof of last theorem.

Let G be an arbitrary locally compact Housdorff group and μ be a right invariant Haar measure on G . The space $L^1(G)$ of integrable functions on G , with the convolution taken as product, is a Banach Algebra. P. Civin and B. Yood [4] have shown that; if G is commutative and infinite set then the Banach Algebra $L^1(G)$ is not Arens regular. N.Young [14] has extended this result to non-commutative case. A.Ulger [13] has

presented a very simple proof of the Theorem, which says that the group algebra $L^1(G)$ is Arens regular if and only if G is finite. In this paper we present a Theorem, which shows that the Young's result does not hold in a hypergroup. This theorem is an application of Theorem 2.2.

Theorem 2.4. There is a hypergroup algebra $M(X)$ which has regular multiplication and $L(X)$ is Arens regular but X is not finite.

Construction. Let $A=[a_1, b_1]$, $B=[a_2, b_2]$, $C=\{a, b\}$ be subsets of an ordered set in the ordered topology and A, B, C are disjoint. Let $X=\{e\} \cup A \cup B \cup C \cup \{z\}$, with the topology in which e, a, b, z are isolated points and supposed $\phi: X \rightarrow [0, 1]$ is a continuous function with $\text{supp } \phi = X$. Define $\lambda: A \times B \rightarrow M_p(C)$ by

$$\lambda_{(x,y)} = \phi(x)\phi(y)\delta_a + (1-\phi(x)\phi(y))\delta_b.$$

So, $\lambda_{(x,y)} = \lambda_{(y,x)}$. If $f \in C_0(X) (-C_c(X))$ then,

$$\lambda_{(x,y)}(f) = \int_X f(t) d\lambda_{(x,y)}(t) = \phi(x)\phi(y)f(a) + (1-\phi(x)\phi(y))f(b)$$

Hence, if $\{(x_n, y_n)\}$ is sequence converge to (x, y) then

$$\lim_n \lambda_{(x_n, y_n)}(f) = \lambda_{(x,y)}(f).$$

Therefore, $\lambda: A \times B \rightarrow (M_p(C), \text{Weak}^*)$ is continuous. By Theorem 2.2 X is a hypergroup.

To prove $M(A \cup B) \subseteq L(X)$, let $\mu \in M(A \cup B)$. Then, $\mu = \mu_A + \mu_B$. Suppose that $x \in B$, then,

$$\begin{aligned} \delta_x |\mu_A| &= \int_X \int_X \lambda_{(u,v)} d\delta_x(v) d|\mu_A|(u) \\ &= \int_X \lambda_{(u,x)} d|\mu_A|(u), \end{aligned}$$

So, for $x, y \in B$,

$$\begin{aligned} \|\delta_x |\mu_A| - \delta_y |\mu_A|\| &= \left\| \int_X \lambda_{(u,x)} - \lambda_{(u,y)} d|\mu_A|(u) \right\| \\ &= \|\phi(x) - \phi(y)\| \left\| \int_X \phi(u) d|\mu_A|(u) \delta_a - \int_X \phi(u) d|\mu_A|(u) \delta_b \right\| \\ &\leq 2\|\phi(x) - \phi(y)\| \|\mu_A\|(\phi), \end{aligned}$$

ϕ is continuous, so $\mu_A \in L(X)$. Similarly, $\mu_B \in L(X)$. Then $L(X) = M(X - \{e\})$.

We now prove that $\phi: M(A) \times M(B) \rightarrow M(C)$ is Arens regular. First suppose that $\mu_A \in M(A)$, $\nu_B \in M(B)$. So,

$$\begin{aligned} \phi(\mu_A, \nu_B) &= \mu_A * \nu_B = \int_X \int_X \lambda_{(x,y)} d\mu_A(x) d\nu_B(y) \\ &= \int_X \int_X [\phi(x)\phi(y)\delta_a + (1-\phi(x)\phi(y)\delta_b)] d\mu_A(x) d\nu_B(y) \\ &= \mu_A(\phi)\nu_B(\phi)\delta_a + (\mu_A(1)\nu_B(1) - \mu_A(\phi)\nu_B(\phi))\delta_b. \end{aligned}$$

Now, suppose $\{(\mu_A)_n\}, \{(\nu_B)_m\}$ are two sequences such that

$w^* - \lim_n w^* - \lim_m \phi((\mu_A)_n, (\nu_B)_m), w^* - \lim_m w^* - \lim_n \phi((\mu_A)_n, (\nu_B)_m)$ exist, then

$$\begin{aligned} w^* - \lim_n w^* - \lim_m \phi((\mu_A)_n, (\nu_B)_m) &= w^* - \lim_n w^* - \lim_m [(\mu_A)_n(\phi)(\nu_B)_m(\phi)\delta_a \\ &\quad + ((\mu_A)_n(I)(\phi)_m(1) - (\mu_A)_n(\phi)(\nu_B)_m(\phi)\delta_b)] \\ &= w^* - \lim_m w^* - \lim_n \phi((\mu_A)_n, (\nu_B)_m). \end{aligned}$$

Hence, ϕ is Arens regular. By Theorem 2.3, $L(X)$ is Arens regular, but X is not finite.

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