Characterization of Filters Preserving Reciprocality

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ABSTRACT

In this paper we characterize the system function of a linear filter that its output will be a reciprocal process whenever its input is a reciprocal one.

Introduction

Let $X = \{X(t), -\infty < t < \infty\}$ be a process defined on some complete probability space $(\Omega, \mathcal{F}, P)$. The notion of reciprocality was first defined by Jamison [1], and studied in some extend by pasha [2] and [3]. The process X has reciprocal property on $(-\infty, \infty)$ if for each $n \in \mathbf{N}$, and for each reals $u < v$, and for each reals $t_1, \ldots, t_n$ in the complement of interval (u, v), and finally for each $t \in (u, v)$, the conditional distribution of X_t given $X_u, X_v, X_{t_1}, \ldots, X_{t_n}$ is the same as the conditional distribution of X_t given X_u and X_v.

In [2] a martingale representation of Gaussian stationary reciprocal processes is given. In [3] the notion of reciprocality is generalized. Jamison [1] proved that the covariance function of Gaussian stationary reciprocal processes with zero mean is of the following form

$$C_X(t) = E(X(s)X(t+s)) = be^{-a|t|} \quad t \in \mathbb{R},$$

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for some positive numbers a and b. It is clear that $\sigma^2(X(t)) = b$.

In this paper we make the following assumptions:

Assumption A: we assume that the process X satisfies the following conditions:

(i) X is Gaussion,

(ii) X is stationary,

(iii) The mean of X_t is zero

(iv) The covariance function of X_t is continuous,

(v) X has reciprocal property on $(-\infty, \infty)$.

Linear filters

Let X be the input of a linear filter with quasi system function h, i.e.

$$h(t) = 0, \quad t \leq 0.$$

Let $Y = \{y(t), -\infty < t < \infty\}$ be the output of the system, i.e.

$$Y(t) = \int_0^\infty h(t)X(t-s)ds.$$

It is well known that if the process X is Gaussian and stationary then the out-put process Y also is Gaussian and stationary. In the following we want to determine the function h so that if X statisfies assumption A, then Y satisfies the assumption A, specifically it has reciprocal property.

If X is stationary then the covariance function of Y is given by

$$\begin{aligned} C_Y(t) &= E(y(t+s)y(s)) \\ &= \int_0^\infty h(s)C_X(s+t)ds \\ &= C_X(t) * h(-s) \end{aligned}$$

where $*$ stands for the convolution of the function $C_X(t)$ and $h_1(t) = h(-t)$.

We will use the following notions in the sequel:

$$\begin{aligned} C_X(t) &= E(X(t+s)X(s)), \\ C_Y(t) &= E(Y(t+s)Y(s)) \\ C_{XY}(t) &= E(X(t+s)Y(s)) \\ S_X(w) &= \int_{-\infty}^\infty e^{-itw}C_X(t)dt, \\ H(w) &= \int_0^\infty e^{-itw}h(t)dt. \end{aligned}$$

$S_Y(w)$,$S_{XY}(w)$will be defined similarly.

Lemma. Let X satisfies assumption A ((i)-(iv)), then

$$\begin{aligned} S_{XY}(w) &= S_X(w)H(-w) \\ S_Y(w) &= S_{XY}(w)H(w). \end{aligned}$$

Proof. We have

$$\begin{aligned} C_{XY} &= \int_0^\infty h(s)C_X(s+t)ds \\ &= \int_{-\infty}^\infty h(s)C_X(s+t)ds \\ &= \int_{-\infty}^\infty h(-s)C_X(t-s)ds \\ &= (C_X * h_1)(t) \end{aligned}$$

where $h_1(t) = h(-t)$. Therefore by taking the Fourier transform we will have

$$\begin{aligned} S_{XY}(w) &= S_X(w).H_1(w) \\ &= S_X(w)H(-w). \end{aligned}$$

Where $H_1(w)$ is the fourier transform of h_1, which is equal to $H(-w)$. A similar computation will prove the second equality.

Now we have the following theorem.

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Theorem 1. Let X satisfies assumption A ((i)-(v)) and $C_X(t) = be^{-a|t|}$. Let Y be the output of the linear quasi system with system function h. Then Y is reciprocal if and only if

$$H(w)H(-w) = \frac{a'b'(a^2+w^2)}{ab(a'^2+w^2)}$$

for some positive numbers a, b, a', b'.

Proof. Assume that the input and output of the system satisfies assumption A. From

$$C_X(t) = be^{-a|t|}$$

we get

$$S_X(w) = \frac{2ab}{a^2+w^2}.$$

Similarly, for some $a' > 0$, $b' > 0$, we have

$$C_Y(t) = b'e^{-a'|t|},$$

Therefore

$$S_Y(w) = \frac{2a'b'}{a'^2+w^2}$$

But, from lemma 1, we have

$$\begin{aligned} S_Y(w) &= S_{XY}(w)H(w) \\ &= S_{XY}(w).H(-w)H(w) \end{aligned}$$

Therefore

$$\frac{2a'b'}{a'^2+w^2} = \frac{2ab}{a^2+w^2}H(w)H(-w)$$

From here we get

$$H(w)H(-w) = \frac{a'b'(a^2+w^2)}{ab(a'^2+w^2)}.$$

Now assume that H satisfies the above relation and the input process satisfies assumption A ((i)-(v)), we prove that Y satisfies assumption A ((i)-(v)). The only property that we have to prove is the reciprocal property of Y. From lemma 1 and the given condition on H we have

$$\begin{aligned} S_Y(w) &= S_X(w)H(w).H(-w) \\ &= \frac{2ab}{a^2+w^2}.\frac{a'b'(a^2+w^2)}{ab(a'^2+w^2)}. \end{aligned}$$

Thus,

$$S_Y(w) = \frac{2a'b'}{a'^2+w^2}.$$

This is the Fourier transform of a function of the following form

$$C_Y(t) = b'e^{-a'|t|}$$

Now from the Jamison result in [1] we conclude that Y has reciprocal propertly.

Example: An example of this kind of filters is

$$h(t) = \frac{b'}{b't^2 + b\pi^2},$$

The Fourier transform of h is

$$H(w) = \sqrt{\frac{b'}{b}}e^{-\pi w\sqrt{\frac{b}{b'}}}$$

Therefore, for any $a > 0$ we have

$$\begin{aligned} H(w)H(-w) &= \frac{b'}{b} \\ &= \frac{b'a(a^2+w^2)}{ba(a^2+w^2)}. \end{aligned}$$

This filter will take an input with covariance function

$$C_X(t) = be^{-a|t|}$$

to an output with covariance function

$$C_Y(t) = b'e^{-a|t|}.$$

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This filter gives more weight to the most recent input than to the most far inputs.

References

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