

# Smoothness and Rotundity in Banach Spaces

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## Abstract

The concept of rotundity is not far from differentiability. There are several papers in the literature devoted to the study of relations between rotundity and smoothness in Banach spaces. In this paper, we study new relations between some kinds of rotundity and smoothness in Banach spaces. In particular, we investigate relations between one kind of rotundity, which is called strongly very rotund, and very smoothness, in Banach spaces. A Banach space is rotund if the midpoint of every two distinct points of unit sphere is in the open unit ball of the Banach space. A Banach space  $X$  is smooth if its norm is Gateaux differentiable at every non-zero point of the space and it is very smooth if the norm is very Gateaux differentiable, that is, the second dual norm in the second dual of  $X$  is Gateaux differentiable at every non zero point of  $\hat{X}$ .

## Introduction

Let  $X$  be a real Banach space and  $\|\cdot\|$  be a norm on  $X$ . For  $0 \neq x \in X$  we define  $D_x = \{f \in X^*: f(x) = \|x\| \|f\|^*, \|f\|^* = \|x\|\}$ , which is non-empty by the Hahn-Banach Theorem.

**Definition1.** The norm of  $X$  is Gateaux differentiable at  $0 \neq x_0 \in X$  if there exists  $f \in X^*$  such that

$$\lim_{t \rightarrow 0} \frac{\|x_0 + th\| - \|x_0\|}{t} = f(h)$$

for all  $h \in X$ .

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Let  $0 \neq x \in X$ . It is obvious that the norm is Gateaux differentiable at  $x$  if and only if the norm is Gateaux differentiable at  $\frac{x}{\|x\|}$ . It is proved in [1, page 5] that the norm is Gateaux differentiable at  $\frac{x}{\|x\|}$  if and only if there is a unique  $f \in S_{X^*}$  such that  $f\left(\frac{x}{\|x\|}\right) = 1$  and hence the norm is Gateaux differentiable at  $0 \neq x$  if and only if  $D_x$  contains only one point.

We say that the norm is smooth if the norm is Gateaux differentiable at all  $0 \neq x \in X$ .

**Definition2.** The norm of  $X$  is very Gateaux differentiable at  $0 \neq x \in X$  if the norm of  $X^{**}$  is Gateaux differentiable at  $\hat{x}$ , where  $\hat{x}(f) = f(x)$  for every  $f \in X^*$ .

We say that the norm is very smooth if the norm is very Gateaux differentiable at all  $0 \neq x \in X$ .

**Definition3.** The norm of  $X$  is Frechet differentiable at  $0 \neq x_0 \in X$  if the norm is Gateaux differentiable at  $x_0$  and the above limit is uniform for  $h \in S_X$ , or equivalently (Smulyan Lemma, [2, Lemma 8.4]), if  $\{f_n\}_{n \in \mathbb{N}}$  and  $\{g_n\}_{n \in \mathbb{N}}$  are two sequences in  $X^*$  such that  $\lim_{n \rightarrow \infty} \|f_n\|^* = \lim_{n \rightarrow \infty} \|g_n\|^* = \|x_0\|$  and

$$\lim_{n \rightarrow \infty} f_n(x_0) = \lim_{n \rightarrow \infty} g_n(x_0) = \|x_0\|^2, \text{ then } \lim_{n \rightarrow \infty} \|f_n - g_n\|^* = 0.$$

Obviously very Gateaux differentiability implies Gateaux differentiability. It is shown in [6, Theorem3] that Frechet differentiability implies very Gateaux differentiability.

**Definition4.** The norm of  $X$  is rotund (R) if for  $x, y \in S_X$  such that  $\|\frac{x+y}{2}\| = 1$ , we have  $x = y$ . Geometrically this means that  $S_X$  has no non-trivial line segment. A Banach space with this property is called rotund.

**Definition5.** The norm of  $X$  is very rotund (VR) if for  $0 \neq x \in X, x^{**} \in X^{**}$ , and  $f \in D_x$  such that  $\|x^{**}\|^{**} = \|x\|$ ,  $x^{**}(f) = \|x\|^2$  we have  $x^{**} = \hat{x}$ .

If we divide  $x, x^{**}, f$  by  $\|x\|$  we can see that this definition is equivalent to say that for  $x \in S_X, x^{**} \in S_{X^{**}}$  and  $f \in D_x$ , relation  $x^{**}(f) = 1$ , implies  $x^{**} = \hat{x}$ .

**Definition6.** The norm of  $X$  is strongly very rotund (SVR) if for  $0 \neq x \in X, x^{(4)} \in X^{(4)}$  and  $f \in D_x$  such that  $\|x^{(4)}\|^{****} = \|x\|$ ,  $f(x) = x^{(4)}(\hat{f}) = \|x\|^2$  we have  $x^{(4)} = \hat{\hat{x}}$ , where  $X^{(4)}$  is the fourth dual of  $X$  and  $\hat{f}(x^{**}) = x^{**}(f)$  for every  $x^{**} \in X^{**}$ .

If we divide  $x, x^{(4)}, f$  by  $\|x\|$  we can see that this definition is equivalent to say that for  $x \in S_X, x^{(4)} \in S_{X^{(4)}}$  and  $f \in D_x$ , relation  $x^{(4)}(\hat{f}) = 1$ , implies  $x^{(4)} = \hat{x}$ .

**Definition 7.** The norm of  $X$  is locally uniformly rotund (LUR) if for  $x \in S_X$  and any sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $B_X$  such that  $\lim_{n \rightarrow \infty} \left\| \frac{x_n + x}{2} \right\| = 1$ , we have  $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ . In this definition  $\lim_{n \rightarrow \infty} \left\| \frac{x_n + x}{2} \right\| = 1$  implies that  $\|x_n\| \rightarrow 1$ . Hence if  $\lim_{n \rightarrow \infty} \left\| \frac{x_n + x}{2} \right\| = 1$ , we have  $\left\| \frac{x_n + x}{2} \right\| \rightarrow 1$  and if  $\left\| \frac{x_n}{\|x_n\|} - x \right\| \rightarrow 0$  we have  $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ .

Therefore, this definition is equivalent to say that for  $x \in S_X$  and any sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $S_X$  such that  $\lim_{n \rightarrow \infty} \left\| \frac{x_n + x}{2} \right\| = 1$ , we have  $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$ .

**Example 8.** Let  $X = \ell_2$  and  $\|\cdot\|_2$  be the canonical norm on  $X$ . By parallelogram law in the Hilbert space  $\ell_2$ , we have  $\|x + y\|_2^2 + \|x - y\|_2^2 = 2\|x\|_2^2 + 2\|y\|_2^2$  for every  $x, y \in X$ . Let  $x \in S_X$  and  $\{x_n\}_{n \in \mathbb{N}} \subseteq S_X$  such that  $\lim_{n \rightarrow \infty} \left\| \frac{x_n + x}{2} \right\|_2 = 1$ . Since  $\|x_n - x\|_2^2 = 2\|x_n\|_2^2 + 2\|x\|_2^2 - \|x_n + x\|_2^2 = 4 - \|x_n + x\|_2^2$ , we have  $\|x_n - x\|_2 \rightarrow 0$ . Therefore,  $\|\cdot\|_2$  is LUR on  $X$ .

**Definition 9.** The norm of  $X$  is weakly locally uniformly rotund (WLUR) if for  $x \in S_X$  and any sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $B_X$  such that  $\lim_{n \rightarrow \infty} \left\| \frac{x_n + x}{2} \right\| = 1$ , we have  $\lim_{n \rightarrow \infty} f(x_n - x) = 0$ , for each  $f \in X^*$ .

**Definition 10.** The norm of  $X$  is weakly uniformly rotund (WUR) if for any two sequence  $\{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}$  in  $B_X$  such that  $\lim_{n \rightarrow \infty} \left\| \frac{x_n + y_n}{2} \right\| = 1$ , we have  $\lim_{n \rightarrow \infty} f(x_n - y_n) = 0$ , for each  $f \in X^*$ .

By the same method applied after Definition 7, it is easy to prove that  $B_X$  can be replaced by  $S_X$  in the Definitions 9 and 10.

Let the norm of  $X$  be SVR and  $x \in S_X, x^{**} \in S_{X^{**}}, f \in D_x$  such that  $x^{**}(f) = 1$ . We have  $\widehat{x^{**}} \in X^{(4)}$  where  $\widehat{x^{**}}(F) = F(x^{**})$  for every  $F \in X^{***}$ , and  $\|\widehat{x^{**}}\|^{****} = \|x\| = 1$ , and  $f(x) = \widehat{x^{**}}(\hat{f}) = \|x\|^2 = 1$ , then  $\widehat{x^{**}} = \hat{x}$  which implies that  $x^{**} = \hat{x}$ . Therefore, the norm of  $X$  is VR. Let the norm of  $X$  be VR and  $x, y \in S_X$  such that  $\left\| \frac{x+y}{2} \right\| = 1$ . By the Hahn-Banach Theorem, there exists  $f \in S_{X^*}$  such that  $f\left(\frac{x+y}{2}\right) = 1$ , then we have  $f(x) = f(y) = 1$ . Since  $f \in D_x, \hat{y} \in S_{X^{**}}$  and  $\hat{y}(f) = f(y) = f(x) = 1$ , we have  $\hat{y} = \hat{x}$  and

so  $x = y$ . Hence the norm of  $X$  is R. Consequently we have:

$$\text{SVR} \rightarrow \text{VR} \rightarrow \text{R}.$$

We will prove in the next section that in reflexive spaces the properties R, VR and SVR are equivalent.

Obviously we have the following implications:

$$\text{LUR} \rightarrow \text{WLUR} \rightarrow \text{R},$$

$$\text{WUR} \rightarrow \text{WLUR} \rightarrow \text{R}.$$

The converse of the above implications is not true in general. To see this, consider the following example:

**Example 11.** Let  $X = \ell_2$ ,  $x = (x_1, x_2, \dots) \in X$ ,  $x' = (0, x_2, x_3, \dots)$  and  $\|x\|_F = |x_1| + \|x'\|_2$ . This is an equivalent norm on  $X$ . Let  $\{\alpha_n\}_{n \in \mathbb{N}}$  be a decreasing sequence of positive numbers converging to zero. Let  $T: \ell_2 \rightarrow \ell_2$ , where  $T(x_1, x_2, \dots) = (\alpha_1 x_1, \alpha_2 x_2, \dots)$ . This is a continuous linear map. For  $x \in \ell_2$  define:

$$\|x\|_A^2 = \|x\|_F^2 + \|T(x)\|_2^2.$$

It is easy to check that this is an equivalent norm on  $\ell_2$ . Moreover, it is shown in [5, Example 3] that  $\|x\|_A$  is R norm on  $\ell_2$ . Since  $\ell_2$  is reflexive, this norm is VR and SVR but it is shown in [7, Example] that this norm is not WLUR.

In [2], [6], [7] and [8] relations between rotundity and smoothness are investigated. We recall some of them:

**Theorem 12.** [6, Lemma 1(2)] If  $\|\cdot\|^*$  is smooth on  $X^*$  then  $\|\cdot\|$  is VR on  $X$ .

**Theorem 13.** [7, Theorem 1(3)] If  $\|\cdot\|$  is LUR on  $X$  then  $\|\cdot\|^*$  is Frechet differentiable on  $D_X = \bigcup_{x \in X} D_x$ .

In the next section we investigate some other relations between rotundity and smoothness.

## Results

**Theorem 1.** If  $X$  is a reflexive space then R, VR and SVR are equivalent properties in  $X$ .

**Proof.** It suffices to show that R implies SVR. Let  $\|\cdot\|$  be R norm on  $X$  and  $x \in X, x^{(4)} \in X^{(4)}$

and  $f \in X^*$  such that  $\|x^{(4)}\|^{****} = \|x\| = \|f\|$ ,  $f(x) = x^{(4)}(\hat{f}) = \|x\|^2$ . Since  $X$  is reflexive we have  $\hat{\hat{X}} = X^{(4)}$ . Hence there exists  $y \in X$  such that  $\hat{y} = x^{(4)}$  and  $\|x^{(4)}\|^{****} = \|y\| = \|x\|$ . Since  $f(y) = \hat{y}(\hat{f}) = x^{(4)}(\hat{f}) = \|x\|^2 = f(x)$  we have  $f\left(\frac{x+y}{2}\right) = \|x\|^2$  which implies that  $\left\|\frac{x+y}{2}\right\| = \|x\|$  and hence  $\left\|\frac{\frac{x}{\|x\|} + \frac{y}{\|y\|}}{2}\right\| = 1$ . Since  $\|\cdot\|$  is R we have  $\frac{x}{\|x\|} = \frac{y}{\|y\|}$ , which implies that  $x = y$  and so  $\hat{x} = \hat{y} = x^{(4)}$ . Consequently  $\|\cdot\|$  is SVR.

**Theorem2.** The norm  $\|\cdot\|$  is VR on  $X$  if and only if  $\|\cdot\|^*$  is Gateaux differentiable on  $D_X = \bigcup_{x \in X} D_x$

**Proof.** Let  $\|\cdot\|$  on  $X$  be VR and  $f \in D_X$ . There exists  $x \in X$  such that  $f \in D_x$ . Since  $\hat{x}(f) = \|x\| \|f\|^*$ ,  $\|\hat{x}\|^{**} = \|f\|^*$  we have  $\hat{x} \in D_f$ . If  $F \in D_f$  then  $F(f) = \hat{x}(f) = \|x\|^2$  and so  $F = \hat{x}$ . Consequently  $D_f$  contains only  $\hat{x}$  and hence  $\|\cdot\|^*$  is Gateaux differentiable at  $f$ .

Conversely let  $\|\cdot\|^*$  be Gateaux differentiable on  $D_X = \bigcup_{x \in X} D_x$  and  $f \in D_x$  for some  $x \in X$ . Let  $X^{**}$  such that  $f(x) = F(f)$ ,  $\|F\|^{**} = \|x\|$ . By Gateaux differentiability  $\hat{x} = F$ . Then  $\|\cdot\|$  on  $X$  is VR.

Let  $\|\cdot\|$  on  $X$  be R and consider  $x \in X$ . If  $f_x \in D_x$  then  $\hat{x}(f_x) = \|x\| \|f_x\|^*$ ,  $\|\hat{x}\|^{**} = \|f_x\|^*$  and consequently  $\hat{x} \in D_{f_x}$ . If  $\hat{y} \in \hat{X}$  such that  $\hat{y}(f_x) = \|\hat{y}\|^{**} \|f_x\|^*$  and  $\|\hat{y}\|^{**} = \|f_x\|^*$  then  $\|x+y\| = \|\hat{x} + \hat{y}\|^{**} = 2 \|f_x\|^*$  that implies  $\hat{x} = \hat{y}$  since  $\|\cdot\|$  is R. Therefore,  $D_{f_x} \cap \hat{X} = \{\hat{x}\}$ . However, we show in the next example that  $D_{f_x} \cap X^{**}$  may contain more than one point, in fact,  $\|\cdot\|^*$  may not be Gateaux differentiable at  $f_x$ . Hence we cannot replace VR by R in Theorem2.

**Example3.** Let  $X = \ell_\infty$  and  $Y = c_0$  be a subset of  $X$  consisting of all sequences converging to zero. Then we have  $Y^* = \ell_1$  [2, Proposition 2.14] and  $X = Y^{**}$  [2, Proposition 2.15]. Let  $\{f_n\}_{n \in \mathbb{N}}$  be a dense subset of  $S_{Y^*}$ . For  $x \in X$  define  $|x|^2 = \|x\|_\infty^2 + \sum_{i=1}^\infty 2^{-i} f_i^2(x)$ . By use of induction we can easily prove that  $|\cdot|$  is a norm, since  $|\cdot|_i$  defined by  $|x|_i^2 = 2^{-i} f_i^2(x)$  is a norm for each  $i \in \mathbb{N}$ . The norm  $|\cdot|$  is an equivalent norm on  $Y$  since  $\|x\|_\infty^2 \leq |x|^2 \leq 2 \|x\|_\infty^2$ . It is proved in [1, page 66 Corollary 6.9(ii)] that  $|\cdot|$  is WUR norm on  $Y$  and we will prove in next theorem

that if the norm  $|\cdot|$  is WUR on  $Y$  then  $|\cdot|^{**}$  is R on  $Y^{**}$ , but it is proved in [3] that there exists  $f \in S_{Y^*}$  such that  $|\cdot|^{***}$  is not Gateaux differentiable at  $\hat{f} \in S_{Y^{***}}$ . If  $F \in X$  such that  $F \in D_f$  then  $\hat{f} \in D_F$ , which implies that  $\hat{f} \in D_X$ . Hence  $|\cdot|^{***}$  is not Gateaux differentiable on  $D_X$ .

**Remark4.** The norm  $|\cdot|^{**}$  is R on  $\ell_\infty$  but it is not VR, by Theorem2 and the last assertion in Example3.

**Theorem5.** If  $\|\cdot\|$  is WUR on  $X$  then  $\|\cdot\|^{**}$  is R on  $X^{**}$ .

**Proof.** Let  $x^{**}, y^{**} \in S_{X^{**}}$  such that  $\|\frac{x^{**}+y^{**}}{2}\|^{**} = 1$ . By Goldstine Theorem [2, Theorem 3.27] there exist a directed set  $\Gamma$  and two nets  $\{x_\alpha\}_{\alpha \in \Gamma}, \{y_\alpha\}_{\alpha \in \Gamma} \subseteq B_X$  such that  $\hat{x}_\alpha \rightarrow x^{**}$  and  $\hat{y}_\alpha \rightarrow y^{**}$  in  $w^*$ -topology. Since  $\|\cdot\|^{**}$  is  $w^*$ -lower semicontinuous [2, Lemma 8.8], for each  $\varepsilon > 0$  there exists  $\alpha_\varepsilon \in \Gamma$  such that

$$\|\hat{x}_\alpha + \hat{y}_\alpha\|^{**} > \|x^{**} + y^{**}\|^{**} - \varepsilon = 2 - \varepsilon$$

for every  $\alpha > \alpha_\varepsilon$ . From  $\|\hat{x}_\alpha + \hat{y}_\alpha\|^{**} \leq 2$ , it follows that  $\|x_\alpha + y_\alpha\| = \|\hat{x}_\alpha + \hat{y}_\alpha\|^{**} \rightarrow 2$ . Since each net contains a sequence as a subnet, there exists a sequence  $\{x_{\alpha_n} + y_{\alpha_n}\}_{n \in \mathbb{N}}$  such that  $\|x_{\alpha_n} + y_{\alpha_n}\| \rightarrow 2$  and since the norm  $\|\cdot\|$  is WUR on  $X$ , we have  $f(x_{\alpha_n} - y_{\alpha_n}) \rightarrow 0$  for every  $f \in X^*$ , which implies that  $\hat{x}_{\alpha_n} - \hat{y}_{\alpha_n} \rightarrow 0$  in  $w^*$ -topology. Therefore,  $x^{**} = y^{**}$  and hence the norm on  $X^{**}$  is R.

**Theorem6.** The norm  $\|\cdot\|$  on  $X$  is SVR if and only if  $\|\cdot\|^*$  is very Gateaux differentiable on  $D_X = \bigcup_{x \in X} D_x$ .

**Proof.** Let  $\|\cdot\|$  on  $X$  be SVR and  $f \in D_X$ . Then there exists  $x \in X$  such that  $f \in D_x$ . Hence  $f(x) = \|f\|^* \|x\| = \|x\|^2$  since  $\|f\|^* = \|x\|$ . Since

$$\hat{\hat{x}}(\hat{f}) = f(x) = \|\hat{\hat{x}}\|^{****} \|\hat{f}\|^{***}, \|\hat{\hat{x}}\|^{****} = \|\hat{f}\|^{***}$$

it follows that  $\hat{\hat{x}} \in D_{\hat{f}}$ . Let  $F \in X^{(4)}$  such that  $F \in D_{\hat{f}}$ . Then

$$F(\hat{f}) = \|\hat{f}\|^{***} \|F\|^{****}, \|F\|^{****} = \|\hat{f}\|^{***} = \|f\|^*$$

Which implies that  $F(\hat{f}) = \|x\|^2 = f(x)$  and so  $F = \hat{\hat{x}}$  by the hypothesis. In fact,  $D_{\hat{f}}$  contains only  $\hat{\hat{x}}$  and consequently  $\|\cdot\|^*$  is very Gateaux differentiable at  $f$ . Conversely, let  $\|\cdot\|^*$  be very Gateaux differentiable on  $D_X = \bigcup_{x \in X} D_x$  and  $f \in D_x$  for some  $x \in X$ .

Let  $F \in X^{(4)}$  such that  $F(\hat{f}) = f(x)$ ,  $\|F\|^{****} = \|\hat{f}\|^{***} = \|f\|^*$ . Then  $F(\hat{f}) = \|\hat{f}\|^{***2}$  or  $F \in D_{\hat{f}}$ . Since  $\hat{x} \in D_{\hat{f}}$  and  $D_{\hat{f}}$  consists of one point by the hypothesis, it follows that  $F = \hat{x}$ , which implies that  $\|\cdot\|$  is SVR on  $X$ .

**Theorem 7.** Let  $X = c_0$ ,  $x = \{x_i\}_{i \in \mathbb{N}} \in X$  and  $\|x\|_r = \|x\|_{\infty} + (\sum_{i=1}^{\infty} 2^{-i} x_i^2)^{1/2}$ . Then  $\|\cdot\|_r$  is an equivalent norm on  $X$  which is WLUR but it is not LUR.

**Proof.** It is easy to see that  $\|\cdot\|_r$  is a norm on  $X$  which is equivalent to  $\|\cdot\|_{\infty}$ . Let  $x \in S_X$ ,  $\{x_n\}_{n \in \mathbb{N}} \subseteq S_X$  such that  $\lim_{n \rightarrow \infty} \|\frac{x_n + x}{2}\|_r = 1$ . For every  $n \in \mathbb{N}$ , let

$$x'_n = \{2^{-\frac{i}{2}}(x_n)_i\}_{i \in \mathbb{N}}, \quad x' = \{2^{-\frac{i}{2}}x_i\}_{i \in \mathbb{N}}, \text{ then } x'_n, x' \in \ell_2. \text{ For every } n \in \mathbb{N}, \text{ let}$$

$$t_n = \|x_n\|_{\infty} + \|x\|_{\infty} - \|x_n + x\|_{\infty},$$

$$t'_n = \|x_n\|_r + \|x\|_r - \|x_n + x\|_r = 2 - \|x_n + x\|_r,$$

$$t''_n = \|x'_n\|_2 + \|x'\|_2 - \|x'_n + x'\|_2.$$

By the hypothesis  $t'_n \rightarrow 0$  and it is obvious that  $t_n \geq 0$  for every  $n \in \mathbb{N}$ .

We have

$$\|x'_n + x'\|_2 \leq \|x'_n\|_2 + \|x'\|_2 = 1 - \|x_n\|_{\infty} + 1 - \|x\|_{\infty} = 2 - \|x_n\|_{\infty} - \|x\|_{\infty},$$

then

$$\begin{aligned} \|x_n + x\|_{\infty} &= \|x_n + x\|_r - \|x'_n + x'\|_2 = 2 - t'_n - \|x'_n + x'\|_2 \\ &\geq 2 - t'_n - 2 + \|x_n\|_{\infty} + \|x\|_{\infty} = \|x_n\|_{\infty} + \|x\|_{\infty} - t'_n. \end{aligned}$$

Hence  $t_n \leq t'_n$  which implies that  $t_n \rightarrow 0$ . Since  $t''_n = t'_n - t_n$ , we have  $t''_n \rightarrow 0$ .

For  $\varepsilon_0 = 1 - \|x\|_{\infty} > 0$  there exists  $m_0 \in \mathbb{N}$  such that  $2 - \varepsilon_0 < \|x_n + x\|_r$  for every  $n > m_0$ . Since  $\|x_n + x\|_r \leq 2\|x_n + x\|_{\infty}$  for every  $n \in \mathbb{N}$ , it follows that

$$\|x_n + x\|_{\infty} > 1 - \frac{\varepsilon_0}{2} \quad (1)$$

for every  $n > m_0$ . Since  $x \in c_0$  there exists  $i_0 \in \mathbb{N}$  such that  $|x_i| < \frac{\|x\|_{\infty}}{2}$  for every  $i > i_0$ ,

which implies that

$$|(x_n + x)_i| = |(x_n)_i + x_i| \leq \|x_n\|_{\infty} + \frac{\|x\|_{\infty}}{2} \quad (2)$$

for every  $i > i_0, n \in \mathbb{N}$ . Since  $t_n \rightarrow 0$ , there exists  $m_1 \in \mathbb{N}$  such that

$$\|x_n\|_{\infty} + \|x\|_{\infty} - \|x_n + x\|_{\infty} < \frac{\|x\|_{\infty}}{2} \quad (3)$$

for every  $n > m_1$ . Set  $m = \max\{m_0, m_1\}$  and let  $n > m$ . If there exists  $i > i_0$  such that

$\|x_n + x\|_{\infty} = |(x_n + x)_i|$ , it follows from (2) and (3) that

$$\frac{\|x\|_\infty}{2} \leq \|x_n\|_\infty + \|x\|_\infty - \|x_n + x\|_\infty < \frac{\|x\|_\infty}{2},$$

which is a contradiction, hence there exists  $i \leq i_0$  such that

$$\|x_n + x\|_\infty = |(x_n + x)_i|. \quad (4)$$

By (1) we have  $|(x_n + x)_i| > 1 - \frac{\varepsilon_0}{2}$ , which implies that

$$2^{\frac{i_0}{2}} \|x'_n\|_2 \geq 2^{\frac{i}{2}} \|x'_n\|_2 \geq |(x_n)_i| > 1 - \frac{\varepsilon_0}{2} - |x_i| \geq 1 - \frac{\varepsilon_0}{2} - \|x\|_\infty = \frac{\varepsilon_0}{2}.$$

Hence  $\|x'_n\|_2 \geq 2^{-\frac{i_0}{2}} \frac{\varepsilon_0}{2}$  for every  $n > m$ . If

$$a = \min \left\{ 2^{-\frac{i_0}{2}} \frac{\varepsilon_0}{2}, \|x'_1\|_2, \|x'_2\|_2, \dots, \|x'_m\|_2 \right\}$$

then  $\|x'_n\|_2 \geq a$  for every  $n \in \mathbb{N}$ . For  $n \in \mathbb{N}$ , if  $\|x'_n\|_2 \leq \|x'\|_2$  then

$$\begin{aligned} 2 &\geq \left\| \frac{x'_n}{\|x'_n\|_2} + \frac{x'}{\|x'\|_2} \right\|_2 \geq \left\| \frac{x'_n}{\|x'_n\|_2} + \frac{x'}{\|x'_n\|_2} \right\|_2 \\ &\quad - \left\| \frac{x'_n}{\|x'_n\|_2} - \frac{x'}{\|x'\|_2} \right\|_2 = \frac{1}{\|x'_n\|_2} \|x'_n + x'\|_2 \\ &\quad - \|x'\|_2 \left( \frac{1}{\|x'_n\|_2} - \frac{1}{\|x'\|_2} \right) = \frac{1}{\|x'_n\|_2} (\|x'_n\|_2 + \|x'\|_2 - t''_n) \\ &\quad - \|x'\|_2 \left( \frac{1}{\|x'_n\|_2} - \frac{1}{\|x'\|_2} \right) = 2 - \frac{t''_n}{\|x'_n\|_2} \end{aligned}$$

and if  $\|x'\|_2 < \|x'_n\|_2$  then

$$\begin{aligned} 2 &\geq \left\| \frac{x'_n}{\|x'_n\|_2} + \frac{x'}{\|x'\|_2} \right\|_2 \geq \left\| \frac{x'_n}{\|x'_n\|_2} + \frac{x'}{\|x'_n\|_2} \right\|_2 \\ &\quad - \left\| \frac{x'_n}{\|x'_n\|_2} - \frac{x'}{\|x'_n\|_2} \right\|_2 = \frac{1}{\|x'_n\|_2} \|x'_n + x'\|_2 \\ &\quad - \|x'_n\|_2 \left( \frac{1}{\|x'_n\|_2} - \frac{1}{\|x'_n\|_2} \right) = \frac{1}{\|x'_n\|_2} (\|x'_n\|_2 + \|x'\|_2 - t''_n) \\ &\quad - \|x'_n\|_2 \left( \frac{1}{\|x'_n\|_2} - \frac{1}{\|x'_n\|_2} \right) = 2 - \frac{t''_n}{\|x'_n\|_2}. \end{aligned}$$

Therefore, we have  $\left\| \frac{x'_n}{\|x'_n\|_2} + \frac{x'}{\|x'\|_2} \right\|_2 \rightarrow 2$  since  $t''_n \rightarrow 0$ . As we shown in Example 8, the

norm  $\|\cdot\|_2$  is LUR, then  $\left\| \frac{x'_n}{\|x'_n\|_2} - \frac{x'}{\|x'\|_2} \right\|_2 \rightarrow 0$  or  $\|x'_n - \frac{\|x'_n\|_2}{\|x'\|_2} x'\|_2 \rightarrow 0$ . Suppose,

without loss of generality, that there exists  $t > 0$  such that  $\|x'_n\|_2 \rightarrow t$ , and hence  $\|x'_n\|_2 \rightarrow \lambda \|x'\|_2$ , where  $\lambda = \frac{t}{\|x'\|_2}$ . For every  $n \in \mathbb{N}$  if  $\lambda_n = \frac{\|x'_n\|_2}{\|x'\|_2}$ , then  $\lambda_n \rightarrow \lambda$  and

$\sum_{i=1}^{\infty} 2^{-i} (x_n - \lambda_n x)_i^2 \rightarrow 0$ , which implies that  $(x_n)_i \rightarrow \lambda x_i$  for every  $i \in \mathbb{N}$ , and hence by

(4) we have  $\|x_n\|_\infty \rightarrow \lambda \|x\|_\infty$ .

Therefore,

$$\|x_n\|_r = \|x_n\|_\infty + \|x'_n\|_2 \rightarrow \lambda \|x\|_\infty + \lambda \|x'\|_2 = \lambda \|x\|_r$$

which implies that  $\lambda = 1$ , since  $\|x_n\|_r = \|x\|_r = 1$ . Therefore,  $\lim_{n \rightarrow \infty} (x_n)_i - x_i = 0$  for every  $i \in \mathbb{N}$ . Moreover,  $|(x_n)_i - x_i| \leq 2$  for every  $n, i \in \mathbb{N}$  since  $x, x_n \in S_X$ .

Let  $0 \neq f \in X^* = \ell_1$ . Then  $f = \{y_i\}_{i \in \mathbb{N}}$ , where  $\sum_{i=1}^{\infty} |y_i| < \infty$ . For  $\varepsilon > 0$ , there exists  $j_0 \in \mathbb{N}$  such that  $\sum_{i=j_0+1}^{\infty} |y_i| < \frac{\varepsilon}{4}$  and  $\alpha = \sum_{i=1}^{j_0} |y_i| > 0$ . On the other hand there exists  $n_0 \in \mathbb{N}$  such that  $|(x_n)_i - x_i| < \frac{\varepsilon}{2\alpha}$  for every  $n > n_0$  and for each  $i \leq j_0$ . For  $n > n_0$  we have

$$|f(x_n - x)| = \left| \sum_{i=1}^{\infty} y_i ((x_n)_i - x_i) \right| \leq \sum_{i=1}^{j_0} |y_i| |(x_n)_i - x_i| + \sum_{i=j_0+1}^{\infty} |y_i| |(x_n)_i - x_i| \leq \sum_{i=1}^{j_0} |y_i| \frac{\varepsilon}{2\alpha} + \frac{\varepsilon}{4} \times 2 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore,  $f(x_n - x) \rightarrow 0$  and hence  $\|\cdot\|_r$  is WLUR on  $X$ .

To show that  $\|\cdot\|_r$  is not LUR, let  $e_n = (t_1, t_2, \dots) \in X, n \in \mathbb{N}$  such that  $t_n = 1$  and  $t_i = 0$  for  $i \neq n$ . Let  $x = e_1, x_n = x + e_n, f \in D_X$ . Then

$$\lim_{n \rightarrow \infty} \|\hat{x}_n\|_r^{**} = \lim_{n \rightarrow \infty} \|x_n\|_r = \|x\|_r = \|\hat{x}\|_r^{**} = \|f\|_r^* = 1 + \frac{1}{\sqrt{2}}.$$

Since  $f \in X^* = \ell_1$ , it has the form  $f = (y_i)_{i \in \mathbb{N}}$ , where  $\sum_{i=1}^{\infty} |y_i| < \infty$ . Hence  $f = \sum_{i=1}^{\infty} y_i e'_i$ , where  $(e'_i)_{i \in \mathbb{N}}$  is the standard base of  $\ell_1$ . It follows that  $f(e_n) = \sum_{i=1}^{\infty} y_i e'_i(e_n) = y_n$  and so  $f(e_n) \rightarrow 0$ . Then

$$\lim_{n \rightarrow \infty} \hat{x}_n(f) = \lim_{n \rightarrow \infty} f(x_n) = f(e_1) = f(x) = \hat{x}(f) = \|x\|_r^2 = (\|f\|_r^*)^2.$$

But

$$\lim_{n \rightarrow \infty} \|\hat{x}_n - \hat{x}\|_r^{**} = \lim_{n \rightarrow \infty} \|x_n - x\|_r = \lim_{n \rightarrow \infty} \|e_n\|_r = \lim_{n \rightarrow \infty} \left(1 + 2^{-\frac{n}{2}}\right) = 1$$

and hence, by the definition  $\|\cdot\|_r^*$  is not Frechet differentiable at  $f \in D_X$ . By Theorem 13,  $\|\cdot\|_r$  is not LUR.

**Corollary 8.** We cannot set WLUR instead of LUR in Theorem 13.

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## References

1. R. Devile, G. Godefroy, V. Zizler, "Smoothness and renorming in Banach spaces, Pitman monographs and surveys in pure and applied mathematics", Longman Scientific and Technical (1993).
2. M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant, V. Zizler, "Functional Analysis and Infinite-dimensional Geometry, CMS Books in Mathematics/ Ouvrages de Mathematiques de la SMC", 8. Springer-Verlag, New York (2001).
3. J. R. Giles, "A non-reflexive Banach space has non-smooth third conjugates space", Canad. Math. Bull, 17 (1) (1974).
5. A. Mark Smith, "Some examples concerning rotundity in Banach spaces", Math. Ann., 233(2) (1978) 155-161.
6. F. Sullivan, "Geometrical properties determined by the higher duals of a Banach space", Illinois Journal of Mathematics, 21(2)(1977) 315-331.
7. AC Yorke, "Differentiability and local rotundity", Journal of the Australian Mathematical Society (Series A), 28(02)(1979) 205-213.
8. Z. H. Zhang, C. J. Zhang, "On very rotund Banach space", Appl. Math. Mech., 21 (8) (2000).