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A Direct Method to Solve Linear two-dimensional Integral Equations using a family of Operational Matrices with block-pulse Functions

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Abstract

An expansion method based on block-pulse functions is developed to find the numerical solution of two-dimensional linear Volterra and Fredholm integral equations of the first and second kind. The present work is based on introducing a family of operational matrices of integration. Error analysis is worked out that shows efficiency and accuracy of the proposed method. Also, some numerical examples are presented.

1. Introduction

Many problems in engineering and mechanics can be transformed into two-dimensional integral equations. For example, it is usually required to solve Fredholm integral equations (FIE) in the calculation of plasma physics [1]. Graham [2] illustrated an application of two-dimensional FIEs in the solution of a problem which arises in electrical engineering. McKee et al. [3] reduced a class of nonlinear telegraph equations to two-dimensional Volterra integral equations (VIE). Some other applications of two-dimensional integral equations can be found in [3, 4].

While the numerical analysis of one-dimensional integral equations is well developed (see, for example, [5-8] and references therein), the numerical methods for two-dimensional integral equations seem to have been discussed in only a few places (see [2-4], [9-17]).

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However, during the last 20 years, significant progress in this area has been made. Brunner and Kauthen introduced in [11] collocation and iterated collocation methods for the solution of two-dimensional linear VIE. They presented an analysis of global and local convergence properties of these methods, and derived results on attainable orders of global convergence and local superconvergence. Brunner extended this study to the case of nonlinear VIE [12].

Important contributions to this field can be also found in the works of Guogiang Han and his co-authors. They obtained asymptotic error expansions for different classical methods, when applied to two-dimensional integral equations, and used them as a basis to introduce extrapolation algorithms. In [13], they used this approach to analyze the solution of linear VFIE by the Nystrom trapezoidal method. In [14] and [15], the iterated collocation method was applied to the solution of nonlinear VIE. Nonlinear FIE have been considered in [16] and [17], where these equations were solved, respectively, by the Nystrom and the iterated Galerkin methods.

Computational methods based on the application of different sets of basis functions have become a very common tool for the solution of different kinds of functional equations, including integral ones. In particular, we are interested here in the use of the block-pulse functions (BPFs).

Of all piecewise constant block functions, the BPF turned out to be the most fundamental and its qualitative as well as quantitative appraisal is presented by Deb et al. [18]. The most striking feature of this function set is its piecewise constant nature. They have been applied as a useful tool in the analysis [19, 20], synthesis [21], identification and other problems of control and systems science [22].

An interesting property of the BPFs, which makes them attractive from the computational point of view, is that the computation of integrals of such functions is very easy. Therefore, when solving numerically differential or integral equations, the use of a basis of BPFs is very advantageous, when compared with other basis of orthogonal functions.

In this paper, we consider the m-set $\{\phi_i(t)\}_{i=0}^{m-1}$ of BPFs on interval [0,1) and introduce a family of operational matrices of integration corresponding to this set so that the operational matrix of integration introduced in [23] is a member of this family. Then, using this family of operational matrices, we derive a family of operational matrices of integration corresponding to the product-set $\{\phi_i(t_1).\phi_j(t_2)\}_{i,j=0}^{m-1}$ of BPFs, defined on $[0,1)\times[0,1)$.

It is shown that the two-dimensional VIEs of the first and second kind can be reduced to lower triangular linear systems of equations that can be solved directly by forward substitution method. The two-dimensional FIEs of the first and second kind is also reduced to linear systems of equations, but in these cases the coefficients matrices of the obtained linear systems are full.

The paper is organized as follows. After an introduction to the present work, a review of block-pulse functions is provided in Section 2. In Section 3, we define a product-set of BPFs and extend the results of Section 2 to this set. Then, in Section 4, we analyze the error representation when a differentiable function is expanded in terms of BPFs and give some bounds on the errors. The numerical method which reduces the two-dimensional Volterra and Fredholm integral equations to linear systems of equations is proposed in Section 5. Some numerical examples are presented in Section 6 to demonstrate the efficiency and accuracy of the method. Finally, in Section 7 we discuss on conditioning of the linear systems obtained in Section 6 and analyze the numerical results reported in the tables.

2. Review of block-pulse functions

Block-pulse functions have been widely used for solving different problems [5, 24]. A complete description of these functions is given in [20, 25]. In this section, we briefly review this class of functions.

2.1. Definition and properties

An m-set of BPFs over the interval [0, T) is defined as

$$\phi_i(t) = \begin{cases} 1, & ih \le t < (i+1)h, \\ 0, & otherwise, \end{cases} \tag{1}$$

where $i = 0,1,...,m-1$, $\mathrm{h} = \mathrm{T/m}$ and m is a positive integer number. The function $\phi_i(t)$ is called the ith BPF.

Without loss of generality consider $T = 1$, so $h = 1/m$. By definition (1), it is clear that

$$\int_0^1 \phi_i(t)dt = h, \qquad i = 0,1,...,m-1. \tag{2}$$

Among various properties of BPFs, the most important ones are disjointness, orthogonality and completeness. From the definition of BPFs, we have

$$\phi_i(t).\phi_j(t) = \begin{cases} \phi_i(t), & i \neq j, \\ 0, & i = j, \end{cases} \tag{3}$$

Where $i, j = 0,1,...,m-1$. This property is known as disjointness of BPFs.

Using (2) and disjointness property, we obtain

$$\int_0^1 \phi_i(t).\phi_j(t)dt = h\delta_{ij}, \tag{4}$$

where $i, j = 0,1,...,m-1$ and δ_{ij} is the Kronecker delta. So, all the BPFs $\phi_i(t)$ are orthogonal to each other.

The other property is completeness. For an arbitrary function f in $L^2([0,1))$, when m approaches infinity, Parseval's identity holds as follows

$$\int_0^1 f^2(t)dt = f_i^2 \left\|\phi_i(t)\right\|^2,$$

where

$$f_i = \frac{1}{h}\int_0^1 f(t)\phi_i(t)dt. \tag{5}$$

2.2. Function approximation and operational matrix

A function f(t) in $L^2([0,1))$ may be expanded in terms of BPFs as

$$f(t) \cong \Phi^t(t)F, \tag{6}$$

where

$$F = [f_0, f_1, ..., f_{m-1}]^t, \tag{7}$$

with f_i s as defined in (5) and

$$\Phi(t) = [\phi_0(t), \phi_1(t), ..., \phi_{m-1}(t)]^t. \tag{8}$$

Now, let ε be a value on (0,1) and approximate $t - ih$, for all values $t \in [ih, (i+1)h)$, by εh. The integral $\int_0^t \phi_i(\tau)d\tau$ can be approximated in terms of BPFs as

$$\int_0^t \phi_i(\tau)d\tau \cong h[0,0,...,\varepsilon,1,...,1]\Phi(t), \tag{9}$$

in which ε is the i th component. Therefore,

$$\int_0^t \Phi(\tau)d\tau \cong P^{(\varepsilon)}\Phi(\tau), \tag{10}$$

where

$$P^{(\varepsilon)} = h\begin{pmatrix} \varepsilon & 1 & 1 & \cdots & 1 \\ 0 & \varepsilon & 1 & \cdots & 1 \\ 0 & 0 & \varepsilon & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \varepsilon \end{pmatrix}_{m\times m}. \tag{11}$$

The upper triangular matrix $P^{(\varepsilon)}$ performs as an integrator and for a function f in $L^2([0,1))$, we have

$$\int_0^t f(\tau)d\tau \cong F^t P^{(\varepsilon)}\Phi(t), \tag{12}$$

where F is the vector of BPF coefficients of function f defined by (7). We call $\{P^{(\varepsilon)} : \varepsilon \in (0,1)\}$ as a family of operational matrices of integration corresponding to the set $\{\phi_i(t)\}_{i=0}^{m-1}$. We note that the operational matrix of integration presented in [23] is a member of this family when $\varepsilon = 1/2$.

3. Operational matrices of integration corresponding to a product-set of block-pulse functions

Let $\{\phi_i(t)\}_{i=0}^{m-1}$ be an m-set of BPFs defined on interval $[0,1)$ and put $D = [0,1)\times[0,1)$. We define the product-set $\{\phi_{ij}(\mathbf{t})\}_{i,j=0}^{m-1}$ of BPFs on D as

$$\phi_{ij}(\mathbf{t}) = \phi_i(t_1).\phi_j(t_2), \tag{13}$$

where $\mathbf{t} = (t_1, t_2)$. Using a similar argument to that employed for the set $\{\phi_i(t)\}$, we can show that the product-set $\{\phi_{ij}(\mathbf{t})\}_{i,j=0}^{m-1}$ is also disjoint, orthogonal and complete. So, any function of two variables $f(\mathbf{t})$ in $L^2(D)$ can be expanded as

$$f(\mathbf{t}) \cong \Psi^t(\mathbf{t})F, \tag{14}$$

where

$$\Psi(\mathbf{t}) = \Phi(t_1)\otimes\Phi(t_2), \tag{15}$$

and

$$F = \left[f_{00}, f_{01},..., f_{0,m-1},..., f_{m-1,0}, f_{m-1,1},..., f_{m-1,m-1}\right]^t. \tag{16}$$

In (15), $\otimes$ denotes the Kronecker product. The block-pulse coefficients f_{ij} in (16) are given by

$$f_{ij} = \frac{1}{h^2}\int_0^1\int_0^1 f(\mathbf{t})\phi_{ij}(\mathbf{t})d\mathbf{t}, \qquad i,j = 0,1,...,m-1, \tag{17}$$

where $d\mathbf{t} = dt_1.dt_2$ and $h = 1/m$.

Similarly, any function $k(\mathbf{t},\mathbf{y})$ in $L^2(D\times D)$ can be expanded in terms of BPFs as

$$k(\mathbf{t},\mathbf{y}) = \Psi^t(\mathbf{t})K\Psi(\mathbf{y}), \tag{18}$$

where K is a block matrix of the form

$$K = [K^{(i,j)}]_{i,j=0}^{m-1}, \tag{19}$$

in which

$$K^{(i,j)} = [k_{iljq}]_{l,q=0}^{m-1}, \qquad i,j = 0,1,...,m-1, \tag{20}$$

and the block-pulse coefficients k_{iljq} are given by

$$k_{iljq} = \frac{1}{h^4}\int_0^1\int_0^1\int_0^1\int_0^1 k(\mathbf{t},\mathbf{y})\phi_{il}(\mathbf{t})\phi_{jq}(\mathbf{y})d\mathbf{t}d\mathbf{y}. \tag{21}$$

By disjointness property of the set $\{\phi_{ij}(\mathbf{t})\}_{i,j=0}^{m-1}$, we obtain

$$\Psi(\mathbf{t})\Psi^t(\mathbf{t}) = diag(\Psi(\mathbf{t})), \qquad \Psi(\mathbf{t})\Psi^t(\mathbf{t})V = \tilde{V}\Psi(\mathbf{t}), \tag{22}$$

where V is a column vector of order m^2 and $\tilde{V} = diag(V)$. Moreover, it can be easily concluded that for any $m^2\times m^2$ matrix B

$$\Psi^t(\mathbf{t})B\Psi(\mathbf{t}) = \hat{B}^t\Psi(\mathbf{t}), \tag{23}$$

where $\hat{B}$ is a column vector whose elements are the diagonal entries of matrix B.

Similar to the one-dimensional case in Section 2, we can obtain a family of operational matrices of integration in order to approximately integrate functions of two variables. For this purpose, let ε be a value on $(0,1)$. Also, for any integrable function f over domain D, consider the notation

$$\int_{\mathbf{0}}^{\mathbf{t}} f(\boldsymbol{\tau})d\boldsymbol{\tau} := \int_0^{t_2}\int_0^{t_1} f(\tau_1,\tau_2)d\tau_1 d\tau_2. \tag{24}$$

By computing $\int_0^{\mathbf{t}}\phi_{ij}(\mathbf{y})d\mathbf{y}$, for $i,j = 0,1,...,m-1$, using the approximate relation (10), we obtain

$$\int_{\mathbf{0}}^{\mathbf{t}}\Psi(\mathbf{y})d\mathbf{y} \cong P_0^{(\varepsilon)}\Psi(\mathbf{t}), \tag{25}$$

where $P_0^{(\varepsilon)}$ is the $m^2\times m^2$ block upper triangular matrix

$$P_0^{(\varepsilon)} = P^{(\varepsilon)}\otimes P^{(\varepsilon)}, \tag{26}$$

in which $P^{(\varepsilon)}$ is the $m\times m$ upper triangular operational matrix defined by (11). Thus $P_0^{(\varepsilon)}$ performs as an integrator and for a function $f(\mathbf{t})$ of two variables, using (14) and (25), we have

$$\int_{\mathbf{0}}^{\mathbf{t}} f(\mathbf{y})d\mathbf{y} \cong F^t P_0^{(\varepsilon)}\Psi(\mathbf{t}), \tag{27}$$

where F is the vector of BPF coefficients of function f defined by (16). We call $\{P_0^{(\varepsilon)} : \varepsilon \in (0,1)\}$ as a family of operational matrices of integration corresponding to the product-set $\{\phi_{ij}(\mathbf{t})\}_{i,j=0}^{m-1}$ of BPFs defined on D.

4. The estimation of the error

In this section, we analyze the error representation when a differentiable function is expanded in terms of BPFs and give some bounds on the error.

Theorem 4.1. Let $f_m(x) = \Phi^t(x)F_m$, be the BPF expansion of the real differentiable function $f(x)$ on $[0,1)$, where $F_m = [f_0, f_1, ..., f_{m-1}]^t$ and f_i s are defined by (5). Suppose, in addition, that there exist a real number M such that

$$\left|f'(x)\right| \le M, \qquad x \in (0,1).$$

Then for all $p \ge 1$

$$\left\|f(x) - f_m(x)\right\|_p = O(1/M). \tag{28}$$

Moreover, if $\bar{F}_m = [\bar{f}_0, \bar{f}_1, ..., \bar{f}_{m-1}]^t$ be an approximation for the BPF coefficients vector F_m and $\bar{f}_m(x) = \Phi^t(x)\bar{F}_m$, then for all $p \ge 1$

$$\left\|f(x) - \bar{f}_m(x)\right\| \le \frac{M}{m} + \left\|F_m - \bar{F}_m\right\|_\infty. \tag{29}$$

Proof. Let $p \ge 1$. From the integral mean value theorem follows

$$\begin{aligned}\left\|f(x) - f_m(x)\right\|_p^p &= \int_0^1 \left|f(x) - f_m(x)\right|^p dx \\ &= \sum_{i=0}^{m-1} \int_{ih}^{(i+1)h} \left|f(x) - f_i\right|^p \\ &= h\sum_{i=0}^{m-1} \left|f(\xi_i) - f_i\right|^p, \quad \xi_i \in (ih, (i+1)h).\end{aligned} \tag{30}$$

From Eq. (5) and integral mean value theorem, we have

$$f_i = \frac{1}{h}\int_{ih}^{(i+1)h} f(x)dx = f(\zeta_i), \qquad \zeta_i \in (ih, (i+1)h). \tag{31}$$

Substituting (31) into (30) gives

$$\begin{aligned}\left\|f(x) - f_m(x)\right\|_p^p &= h\sum_{i=0}^{m-1} \left\|f(\xi_i) - f(\zeta_i)\right\|^p \\ &= h\sum_{i=0}^{m-1} \left|f'(\gamma_i)\right|^p \left|\xi_i - \zeta_i\right|^p \le (Mh)^p,\end{aligned} \tag{32}$$

where we used mean value theorem for function $f(x)$ and γ_i is a number between ξ_i and ζ_i. Taking the p th root from the both sides of (32) gives (28). To prove (29), we write

$$\left\| f(\mathbf{x}) - \bar{f}_m(\mathbf{x}) \right\|_p \le \left\| f(\mathbf{x}) - f_m(\mathbf{x}) \right\|_p + \left\| f_m(\mathbf{x}) - \bar{f}_m(\mathbf{x}) \right\|_p . \tag{33}$$

Also, we have

$$\begin{aligned} \left\| f_m(x) - \bar{f}_m(x) \right\|_p^p &= \int_0^1 \left| f_m(x) - \bar{f}_m(x) \right|^p dx \\ &= \sum_{i=0}^{m-1} \int_{ih}^{(i+1)h} \left| f_i - \bar{f}_i \right|^p dx \\ &= h \sum_{i=0}^{m-1} \left| f_i - \bar{f}_i \right|^p \le \left\| F_m - \bar{F}_m \right\|_\infty^p . \end{aligned}$$

Now, the result can be concluded from the above inequality and (33).

A similar result can be obtained for functions of two variables. First, we formulate the following theorem.

Theorem 4.2. Let $\Omega \subset \Re^2$ be an open convex set, $f : \Omega \to \Re$ be a differentiable function and there exists a real number M such that

$$\left\| f'(\mathbf{x}) \right\| \le M, \qquad \mathbf{x} \in \Omega.$$

Then,

$$\left| f(\mathbf{x}) - f(\mathbf{y}) \right| \le M \left\| \mathbf{x} - \mathbf{y} \right\|_2, \qquad \mathbf{x}, \mathbf{y} \in \Omega. \tag{34}$$

Proof. See [26].

Theorem 4.3. Let $f_m(\mathbf{x}) = \Psi^t(\mathbf{x}) F_m$ be the BPF expansion of the real differentiable function $f(\mathbf{x})$ on $[0,1)\times[0,1)$, where $F_m = [f_{00}, f_{01}, \ldots, f_{0,m-1}, \ldots, f_{m-1,0}, f_{m-1,1}, \ldots, f_{m-1,m-1}]^t$ and f_{ij} s are defined by (17). If $f'(\mathbf{x})$ is bounded on $(0,1)\times(0,1)$, then for all $p \ge 1$

$$\left\| f(\mathbf{x}) - f_m(\mathbf{x}) \right\|_p = O(1/M). \tag{35}$$

Moreover, if $\bar{F}_m = [\bar{f}_{00}, \ldots, \bar{f}_{0,m-1}, \bar{f}_{m-1,0}, \ldots, \bar{f}_{m-1,m-1}]^t$ be an approximation for the BPF coefficients vector F_m and $\bar{f}_m(\mathbf{x}) = \Psi^t(\mathbf{x}) \bar{F}_m$, then for all $p \ge 1$

$$\left\| f(\mathbf{x}) - \bar{f}_m(\mathbf{x}) \right\|_p \le \frac{\sqrt{2}M}{m} + \left\| F_m - \bar{F}_m \right\|_\infty , \tag{36}$$

where M is a real number such that

$$\left\| f'(\mathbf{x}) \right\| \le M, \qquad \mathbf{x} \in (0,1)\times(0,1).$$

Proof. The proof proceeds in complete analogy to the one of Theorem 4.1 and using Theorem 4.2.

We note that the results of Theorem 4.3 can be easily extended for functions with $n > 2$ variables.

5. Numerical method to solve two-dimensional integral Equations

The results obtained in previous sections are used to introduce a direct efficient and simple method to solve two-dimensional Volterra and Fredholm integral equations of the first and second kind.

5.1. Volterra integral equations of the first kind

Consider the following two-dimensional VIE of the first kind

$$f(\mathbf{x}) = \int_0^{\mathbf{x}} k(\mathbf{x},\mathbf{t})u(\mathbf{t})d\mathbf{t}, \tag{37}$$

where $u(\mathbf{x})$ is a real unknown function and $f(\mathbf{x})$ and $k(\mathbf{x},\mathbf{t})$ are given L^2 functions defined on D and $S = \{(x,t) : 0 \le t_i \le x_i < 1, i = 1,2\}$, respectively.

Approximating functions $f(\mathbf{x})$, $u(\mathbf{x})$, and $k(\mathbf{x},\mathbf{t})$ with respect to BPFs, as described by Eqs. (14) and (18), gives

$$\begin{aligned} f(\mathbf{x}) &= \Psi^t(\mathbf{x})F, \\ u(\mathbf{x}) &= \Psi^t(\mathbf{x})U, \\ k(\mathbf{x},\mathbf{t}) &= \Psi^t(\mathbf{x})K\Psi(\mathbf{t}) \end{aligned} \tag{38}$$

where

$$U = [u_{00}, u_{01}, ..., u_{0,m-1}, u_{m-1,0}, u_{m-1,1}, ..., u_{m-1,m-1}]^t, \tag{39}$$

is the vector of BPF coefficients of unknown function $u(\mathbf{x})$. The vector F and matrix K are BPF coefficients of $f(\mathbf{x})$ and $k(\mathbf{x},\mathbf{t})$, respectively and are defined by (16) and (19). Also, $\Psi(\mathbf{x})$ is the vector defined by (15). Our aim is to find an approximation of the solution $u(\mathbf{x})$ by approximating the unknown vector U.

By substituting (38) into (37) and using relations (22) and (23), we obtain

$$\begin{aligned}\Psi^t(\mathbf{x})F &\cong \Psi^t(\mathbf{x})K\int_o^{\mathbf{x}}\Psi(\mathbf{t})\Psi^t(\mathbf{t})U d\mathbf{t}\\ &= \Psi^t(\mathbf{x})K\tilde{U}\int_o^{\mathbf{x}}\Psi(\mathbf{t})d\mathbf{t}\\ &= \Psi^t(\mathbf{x})K\tilde{U}P_0^{(\varepsilon)}\Psi(\mathbf{x})\\ &= \Psi^t(\mathbf{x})\hat{U}^{(\varepsilon)},\end{aligned} \tag{40}$$

where $P_0^{(\varepsilon)}$ is the operational matrix of integration defined by (26), $\tilde{U} = diag(U)$, ε is a known value in $(0,1)$ and $\hat{U}^{(\varepsilon)}$ is an m^2-vector whose elements are the diagonal entries of matrix $K\tilde{U}P_0^{(\varepsilon)}$.

Now, replacing $\cong$ with $=$ in (40) gives

$$\hat{U}^{(\varepsilon)} = F. \tag{41}$$

If we represent $\tilde{U}$ in the block form

$$\tilde{U} = diag(\tilde{U}_0, \tilde{U}_1, ..., \tilde{U}_{m-1}),$$

where

$$\tilde{U}_i = diag(u_{i0}, u_{i1}, ..., u_{i,m-1}),$$

then

$$\hat{U}^{(\varepsilon)} = h\begin{pmatrix} \varepsilon K^{(0,0)}\hat{\tilde{U}}_0 P^{(\varepsilon)} \\ K^{(1,0)}\hat{\tilde{U}}_0 P^{(\varepsilon)} + \varepsilon K^{(1,1)}\hat{\tilde{U}}_1 P^{(\varepsilon)} \\ \vdots \\ K^{(m-1,0)}\hat{\tilde{U}}_0 P^{(\varepsilon)} + ... + K^{(m-1,m-2)}\hat{\tilde{U}}_{m-2} P^{(\varepsilon)} + \varepsilon K^{(m-1,m-1)}\hat{\tilde{U}}_{m-1} P^{(\varepsilon)} \end{pmatrix}, \tag{42}$$

where $K^{(i,j)}\hat{\tilde{U}}_j P^{(\varepsilon)}$ is the column vector of the diagonal entries of matrix $K^{(i,j)}\tilde{U}_j P^{(\varepsilon)}$.

It can be shown that

$$K^{(i,j)}\hat{\tilde{U}}_j P^{(\varepsilon)} = H^{(i,j)}U^{(j)}, \tag{43}$$

where

$$H^{(i,j)} = \begin{pmatrix} \varepsilon k_{i0,j0} & 0 & \cdots & 0 \\ \varepsilon k_{i1,j0} & \varepsilon k_{i1,j1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_{i,m-1,j0} & k_{i,m-1,j1} & \cdots & \varepsilon k_{i,m-1,j,m-1} \end{pmatrix}, \tag{44}$$

and

$$U^{(j)} = [u_{j0}, u_{j1}, ..., u_{j,m-1}]^t. \tag{45}$$

Therefore, equation (41) is a linear system of equations of the form

$$H^{(\varepsilon)}U = F, \tag{46}$$

where

$$H^{(\varepsilon)} = h^2 \begin{pmatrix} \varepsilon H^{(0,0)} & 0 & \cdots & 0 \\ H^{(1,0)} & \varepsilon H^{(1,1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ H^{(m-1,0)} & H^{(m-1,1)} & \cdots & \varepsilon H^{(m-1,m-1)} \end{pmatrix}, \tag{47}$$

and $H^{(i,j)}$, $i = 0,1,...,m-1$, $j = 0,1,...,i$, are defined by (44).

Equation (46) is a lower triangular linear system of equations of order m^2 that can be easily solved by forward substitution method. But, by defining

$$F^{(i)} = [f_{i0}, f_{i1}, ..., f_{i,m-1}]^t, \qquad i = 0,1,...,m-1, \tag{48}$$

we can reduce (46) to m lower triangular linear systems of equations of the form

$$\begin{cases} \varepsilon h^2 H^{(0,0)} U^{(0)} = F^{(0)}, \\ , \\ \varepsilon h^2 H^{(i,i)} U^{(i)} = F^{(i)} - h^2 \sum_{j=0}^{i-1} H^{(i,j)} U^{(j)}, \quad i = 1,2,...,m-1, \end{cases} \tag{49}$$

which are of order m.

The lower triangular linear systems in (49) can be solved by forward substitution method directly and solving the ith system is subject to solving the $(i-1)$th system. So, we can solve the smaller systems in (49) instead of solving the large system (46).

At the end, we note that the coefficient matrix $H^{(\varepsilon)}$, which is of order $m^2 \times m^2$, has only $\frac{m^2(m+1)^2}{4}$ nonzero entries which is very satisfactory from the computational point of view.

5.2. Volterra integral equations of the second kind

Consider the following two-dimensional VIE of the second kind

$$u(\mathbf{x}) = f(\mathbf{x}) + \int_0^{\mathbf{x}} k(\mathbf{x},\mathbf{t})u(\mathbf{t})d\mathbf{t}, \tag{50}$$

where $u(\mathbf{x})$ is a real unknown function and $f(\mathbf{x})$ and $k(\mathbf{x},\mathbf{t})$ are given L^2 functions defined on D and $S = \{(\mathbf{x},\mathbf{t}) : 0 \le t_i \le x_i, i = 1,2\}$, respectively.

Using a similar argument to that employed for the first kind integral equation (37), the integral equation (50) can be reduced to a lower triangular linear system of equations of order m^2 in the form

$$(I - H^{(\varepsilon)})U = F, \tag{51}$$

where I is the identity matrix of order m^2 and F and $H^{(\varepsilon)}$ are defined by (16) and (47), respectively.

The lower triangular system (51) can directly be solved by forward substitution method or one can reduce it to m lower triangular linear systems of equations of the form

$$\begin{cases} \left(I^{(m)} - \varepsilon h^2 H^{(0,0)}\right)U^{(0)} = F^{(0)}, \\ \left(I^{(m)} - \varepsilon h^2 H^{(i,i)}\right)U^{(i)} = F^{(i)} + h^2 \sum_{j=0}^{i-1} H^{(i,j)} U^{(j)}, \quad i = 1,2,...,m-1, \end{cases} \tag{52}$$

where $I^{(m)}$ is the identity matrix of order m and $H^{(i,j)}$ and $U^{(i)}$ are defined by (44) and (45), respectively. Similar to the linear systems in (49), the lower triangular linear systems in (52) also can be solved by forward substitution method directly and solving the ith system, is subject to solving the $(i-1)$th system.

5.3. Fredholm integral equations

In the rest of this section, we consider the following two-dimensional FIEs of the first and second kind

$$f(\mathbf{x}) = \int_0^1 \int_0^{1`} k(\mathbf{x},\mathbf{t})u(\mathbf{t})d\mathbf{t}, \tag{53}$$

and

$$u(\mathbf{x}) = f(\mathbf{x}) + \int_0^1 \int_0^1 k(\mathbf{x},\mathbf{t})u(\mathbf{t})d\mathbf{t}, \tag{54}$$

where $u(\mathbf{x})$ is a real unknown function and $f(\mathbf{x})$ and $k(\mathbf{x},\mathbf{t})$ are given L^2 functions defined on D and $D \times D$, respectively.

From the orthogonality of BPFs follows that

$$\int_0^1 \int_0^1 \Psi(\mathbf{t}).\Psi^t(\mathbf{t})d\mathbf{t} = h^2 I, \tag{55}$$

where I is the identity matrix of order m^2.

By approximating $u(\mathbf{x})$, $f(\mathbf{x})$ and $k(\mathbf{x},\mathbf{t})$ in terms of BPFs, as we described in (38), the integral equations (53) and (54) can be respectively reduced to

$$F \cong h^2 KU, \tag{56}$$

and

$$U \cong F + h^2 KU, \tag{57}$$

where F, K and U are defined by (16), (19) and (39), respectively.

Replacing $\cong$ with $=$ in (56) and (57), respectively gives

$$h^2 KU = F, \tag{58}$$

and

$$(I - h^2 K)U = F. \tag{59}$$

Equations (58) and (59) are linear systems of equations of order m^2 which have full coefficients matrix and can be solved by numerical methods.

In practice, (59) is usually solved by direct Gauss elimination, with associated cost of about $\frac{1}{3}m^6$ multiplications and additions. This algorithm is safe, in the sense that it can not lead to unbounded errors as m is increased, provided that the condition number $Q(m)$ of the coefficient matrix

$$Q(m) = \left\| I - h^2 K \right\| . \left\| \left(I - h^2 K \right)^{-1} \right\|, \tag{60}$$

does not become too large.

For a fixed value m, if h^2 is close to a characteristic value of kernel k, since the coefficient matrix K is an approximation of the kernel k, the condition number $Q(m)$ may indeed be large. But, if k is a bounded kernel such that for a real number M

$$| K(\mathbf{x}, \mathbf{t}) | \leq M, \quad (\mathbf{x}, \mathbf{t}) \in D \times D, \tag{61}$$

then

$$k_{iljq} \leq M, \qquad i, l, j, q = 0,1,...,m-1, \tag{62}$$

and in any natural matrix norm $\|.\|$, we obtain

$$\lim_{m \to \infty} Q(m) = 1. \tag{63}$$

Thus condition number $Q(m)$ is likely to be approximately independent of m and the linear equation (59) is unlikely to be ill-conditioned expect for those values of m that h^2 is close to a characteristic value of kernel k. Therefore, Gauss elimination should cause no problem of accuracy.

For the linear system (58), when m increases the coefficient matrix $h^2 K$ tends to the singular matrix $\mathbf{0}$ and therefore $\left\| (h^2 K)^{-1} \right\|$ is a large number. So, in this case we are faced with an ill-conditioned system of equations and we may have big oscillations in the obtained solutions.

6. Numerical examples

The method presented in this article is applied to some examples. The absolute values of the error at the selected grid points which are proposed as $(x,t)=(l/10,l/10)$, $l=0,1,...,9$, is reported.

The L_2 error and L_2 rate of convergence are defined to be, respectively

$$\| e_m \|_2 = \left(\int_0^1 \int_0^1 | u(\mathbf{x}) - \Psi^t(\mathbf{x}) U |^2 \, d\mathbf{x} \right)^{1/2}, \tag{64}$$

and

$$\rho_m := \log_2 \left(\| e_m \|_2 / \| e_{2m} \|_2 \right), \tag{65}$$

where $u(\mathbf{x})$ is the exact solution of the integral equation and U is the solution of the obtained linear system. For the considered examples, the values of $\| e_m \|_2$ and ρ_m are also inserted in the tables.

All the computations were carried out with Mathematica 7 on a personal computer.

Example 6.1. Consider the following first kind two-dimensional VIE

$$f(x,t) = \int_0^t \int_0^x (xy + te^z) u(y,z) dydz, \qquad 0 \le x,t \le 1,$$

with

$$f(x,t) = x(t + t^2 e^t - te^t) + \frac{1}{3} x^4 (1 - e^{-t}) + \frac{1}{2} x^2 t^2 + \frac{1}{4} x^3 t^2,$$

and the exact solution $u(x,t) = xe^{-t} + t$. The numerical results for this example are shown in Tables 1 and 2. The results in Table 2 show that the method is not applicable to this example, when ε is close to 0. For a detailed description of this issue, see Section 7.

Example 6.2. [27] For the following two-dimensional VIE of the second kind

$$u(x,t) = f(x,t) + \int_0^t \int_0^x (xy^2 + \cos(z)) u(y,z) dydz, \qquad 0 \le x,t \le 1,$$

where

$$f(x,t) = x\sin(t) - \frac{1}{4} x^5 (1 - \cos(t)) - \frac{1}{4} x^2 \sin^2(t),$$

the exact solution is $u(x,t) = x\sin(t)$. Tables 3 and 4 show the numerical results for this example.

Table 1. Numerical results for Example 6.1 when $\varepsilon = 1/2$.

$(x,t) = (l/10, l/10)$	$m = 10$	$m = 20$	$m = 40$	$m = 80$	$m = 160$
$l = 0$	9.2×10^{-2}	4.7×10^{-2}	2.3×10^{-2}	1.2×10^{-2}	5.9×10^{-3}
$l = 1$	6.2×10^{-2}	5.8×10^{-2}	3.3×10^{-2}	1.8×10^{-2}	9.2×10^{-3}
$l = 2$	8.7×10^{-2}	4.7×10^{-2}	2.4×10^{-2}	1.2×10^{-2}	6.0×10^{-2}
$l = 3$	7.0×10^{-2}	3.9×10^{-2}	2×10^{-2}	9.9×10^{-3}	4.9×10^{-3}
$l = 4$	6.7×10^{-2}	3.4×10^{-2}	1.7×10^{-2}	8.6×10^{-3}	4.3×10^{-3}
$l = 5$	6.5×10^{-2}	3.1×10^{-2}	1.6×10^{-2}	7.8×10^{-3}	3.9×10^{-3}
$l = 6$	5.7×10^{-2}	2.9×10^{-2}	1.4×10^{-2}	7.3×10^{-3}	3.6×10^{-3}
$l = 7$	5.7×10^{-2}	2.7×10^{-2}	1.4×10^{-2}	6.9×10^{-3}	3.4×10^{-3}
$l = 8$	5.2×10^{-2}	2.6×10^{-2}	1.3×10^{-2}	6.6×10^{-3}	3.3×10^{-3}
$l = 9$	5.1×10^{-2}	2.6×10^{-2}	1.3×10^{-2}	6.4×10^{-3}	3.2×10^{-3}
$\|e_m\|_2$	3.2×10^{-2}	1.69×10^{-2}	8.69×10^{-3}	4.83×10^{-3}	2.65×10^{-3}
ρ_m	-	0.94	0.92	0.90	0.87

Table 2: L_2 error values corresponding to different values of m and ε for Example 6.1.

$\varepsilon = l/10$	$m = 5$	$m = 10$	$m = 20$	$m = 40$
$l = 1$	$1.66\times10^{+6}$	$9.36\times10^{+14}$	$8.38\times10^{+30}$	$1.31\times10^{+47}$
$l = 2$	$1.47\times10^{+2}$	$2.16\times10^{+7}$	$3.52\times10^{+18}$	$3.07\times10^{+32}$
$l = 3$	$6.55\times10^{+0}$	$7.04\times10^{+2}$	$5.12\times10^{+9}$	$7.28\times10^{+23}$
$l = 4$	5.30×10^{-1}	$2.49\times10^{+0}$	$1.59\times10^{+2}$	$1.57\times10^{+8}$
$l = 5$	9.21×10^{-2}	3.24×10^{-2}	1.69×10^{-2}	8.96×10^{-3}
$l = 6$	1.44×10^{-1}	6.94×10^{-2}	8.49×10^{-2}	3.04×10^{-2}
$l = 7$	1.96×10^{-1}	1.29×10^{-1}	9.82×10^{-2}	8.45×10^{-2}
$l = 8$	2.43×10^{-1}	1.56×10^{-1}	1.12×10^{-1}	5.44×10^{-2}
$l = 9$	2.86×10^{-1}	1.81×10^{-1}	1.25×10^{-1}	6.41×10^{-2}

Example 6.3. Consider the following two-dimensional FIE

$$u(x,t) = f(x,t) + \int_0^1 \int_0^1 (x\sin(t) + ye^z)u(y,z)dydz, \qquad 0 \le x,t \le 1,$$

Where

$$f(x,t) = xe^{-t} + \frac{1}{2}(e^{-1}+1)x.\sin(t) + \frac{1}{2}e - \frac{11}{6}.$$

The exact solution is $u(x,t) = xe^{-t} - 1$. Table 5 illustrates the numerical results for this example.

Table 3: Numerical results for Example 6.2 when $\varepsilon = 1/2$.

$(x,t) = (l/10, l/10)$	$m = 10$	$m = 20$	$m = 40$	$m = 80$	$m = 160$
$l = 0$	2.5×10^{-3}	6.2×10^{-4}	1.6×10^{-4}	3.9×10^{-5}	9.8×10^{-6}
$l = 1$	1.2×10^{-2}	5.6×10^{-3}	2.6×10^{-3}	1.3×10^{-3}	6.3×10^{-4}
$l = 2$	2.2×10^{-2}	1.0×10^{-2}	5.1×10^{-3}	2.5×10^{-3}	1.2×10^{-3}
$l = 3$	3.1×10^{-2}	1.5×10^{-2}	7.4×10^{-3}	3.7×10^{-3}	1.8×10^{-3}
$l = 4$	4.0×10^{-2}	1.9×10^{-2}	9.6×10^{-3}	4.8×10^{-3}	2.4×10^{-3}
$l = 5$	4.8×10^{-2}	2.3×10^{-2}	1.2×10^{-2}	5.8×10^{-3}	2.9×10^{-3}
$l = 6$	5.5×10^{-2}	2.7×10^{-2}	1.3×10^{-2}	6.6×10^{-3}	3.3×10^{-3}
$l = 7$	6.0×10^{-2}	3.0×10^{-2}	1.5×10^{-2}	7.4×10^{-3}	3.7×10^{-3}
$l = 8$	6.5×10^{-2}	3.2×10^{-2}	1.6×10^{-2}	8.0×10^{-3}	4.0×10^{-3}
$l = 9$	6.8×10^{-2}	3.4×10^{-2}	1.7×10^{-2}	8.4×10^{-3}	4.2×10^{-3}
$\|e_m\|_2$	2.07×10^{-2}	1.04×10^{-2}	5.18×10^{-3}	2.59×10^{-3}	1.29×10^{-3}
ρ_m	-	1.00	1.00	1.00	1.00

7. Some comments on the results

We approximated two-dimensional VIEs of the first and second kind to a family of linear systems of equations using the family $\{P^{\varepsilon} : \varepsilon \in (0,1)\}$ of operational matrices. These linear systems were depended on ε. Two obvious questions are:

(i) Is there any ε that is optimum in some sense?

(ii) How is the behavior of solutions when ε varies in the interval $(0,1)$?

These questions are very important, especially when we are faced with the first kind VIEs. This type of equations are in general ill-posed and obtaining their numerical solution often leads to solving a linear system of equations of a large condition number. These equations may have no solution, while if a solution exists the solution is generally unstable, and small changes to the problem can cause a very large change to the answer obtained.

In this section, we study these questions and analyze the numerical results reported in the tables.

To obtain the family $\left\{P^{(\varepsilon)} : \varepsilon \in (0,1)\right\}$ of operational matrices, we have used the following approximations in (9):

$$\int_o^t \phi_i(t)dt = \begin{cases} 0, & t < ih, \\ t - ih, & ih \le t < (i+1)h, \\ h, & t \ge (i+1)h, \end{cases} \cong \begin{cases} 0, & t < ih, \\ \varepsilon h, & ih \le t < (i+1)h, \\ h, & t \ge (i+1)h, \end{cases} \tag{66}$$

where $i = 0,1,...,m-1$. A criterion to specify ε suitably is to solve the $\min-\max$ problems

$$E_i(\varepsilon) = \min_{\varepsilon\in(0,1)} \max_{t\in[ih,(i+1)h]} |t - ih - \varepsilon h|, \qquad i = 0,1,...,m-1. \tag{67}$$

It can be easily shown that

$$E_i = \frac{h}{2}, \qquad i = 0,1,...,m-1. \tag{68}$$

Therefore, $\varepsilon = \frac{1}{2}$ is optimum in the sense that the approximation errors in (10) and (25) are minimum.

Tables 2 and 4 also confirm the superiority of the choice $\varepsilon = \frac{1}{2}$. We explain this by analyzing the behavior of solutions of linear systems (46) and (51) when m is a fixed integer and ε varies from $\frac{1}{2}$ to 0 or to1.

We observe in Tables 2 and 4 that, for a fixed value of m, the L_2 error $\| e_m \|_2$ increases when ε varies from $\frac{1}{2}$ to 0 or to1. This is because the error values $E_i(\varepsilon), i = 0,1,...,m-1,$ in (67) become larger when ε tends to 0 or to1. But the increment in $\| e_m \|_2$ is not the same for VIEs of the first and second kind.

Table 4: L_2 error values corresponding to different values of m and ε for Example 6.2.

$\varepsilon = \frac{l}{10}$	$m = 5$	$m = 10$	$m = 20$	$m = 40$
$l = 1$	4.60×10^{-2}	2.36×10^{-2}	1.20×10^{-2}	6.04×10^{-3}
$l = 2$	4.33×10^{-2}	2.24×10^{-2}	1.13×10^{-2}	5.68×10^{-3}
$l = 3$	4.20×10^{-2}	2.15×10^{-2}	1.08×10^{-2}	5.41×10^{-3}
$l = 4$	4.16×10^{-2}	209×10^{-2}	$1.05\times10^{+2}$	5.24×10^{-3}
$l = 5$	4.14×10^{-2}	2.07×10^{-2}	1.04×10^{-2}	5.18×10^{-3}
$l = 6$	4.21×10^{-2}	2.10×10^{-2}	1.05×10^{-2}	5.24×10^{-3}
$l = 7$	4.68×10^{-2}	2.19×10^{-1}	1.09×10^{-2}	5.43×10^{-3}
$l = 8$	5.18×10^{-2}	2.33×10^{-2}	1.15×10^{-2}	5.74×10^{-3}
$l = 9$	5.24×10^{-2}	2.52×10^{-2}	1.24×10^{-2}	6.14×10^{-3}

Indeed, when ε changes from $\frac{1}{2}$ to zero the matrix $H^{(\varepsilon)}$ tends to the singular matrix $\mathbf{0}$ and thus $\left\|\left(H^{(\varepsilon)}\right)^{-1}\right\|$ may be very large. So, the linear system (46) is an ill-conditioned system and we may have very large changes in the solutions (see Table 2). But, since VIEs of the second kind are well-posed problems, the solution of (51) is stable when ε varies from $\frac{1}{2}$ to zero although the L_2 error $\| e_m \|_2$ increases.

When ε changes from $\frac{1}{2}$ to 1, again the L_2 error $\| e_m \|_2$ increases for both VIEs of the first and second kind but for the first kind problem we again may have oscillations in the solutions because this problem is in general ill-possed (see Tables 2,4).

Therefore, as we concluded mathematically, the numerical results of Tables $1-4$ also show that $\frac{1}{2}$ is the best value for ε.

Now, we analyze the behavior of solutions of linear systems (46) and (51) when ε is a fixed value and m increases. For this purpose, we define

$$Q(\varepsilon,m)=\left\|I-H^{(\varepsilon)}\right\|.\left\|(I-H^{(\varepsilon)})^{-1}\right\|, \qquad m\geq 1, \varepsilon\in(0,1),$$

where $H^{(\varepsilon)}$ is the matrix defined by (47). By the same argument used at the end of Section 5, we can show that

$$\lim_{m\to\infty} Q(\varepsilon,m)=1, \qquad \varepsilon\in(0,1).$$

Therefore, for a fixed value $\varepsilon\in(0,1)$, the condition number $Q(\varepsilon,m)$ is independent of m and the linear system (51) is unlikely to be ill-conditioned (expect for those values of m that h^2 is close to an eigenvalue of matrices $H^{(i,i)}, i=0,1,...,m-1$. But for the linear system (46), $\left\|\left(H^{(\varepsilon)}\right)^{-1}\right\|$ becomes a large number when m increases and we have an ill-conditioned system. So, we may have big oscillations in the solutions (see Table 2).

Table 5: Numerical results for Example 6.3.

$(x,t)=\left(\frac{l}{10},\frac{l}{10}\right)$	$m=10$	$m=20$	$m=40$	$m=80$	$m=160$
$l=0$	2.7×10^{-1}	1.2×10^{-1}	6.2×10^{-2}	3.1×10^{-2}	1.6×10^{-2}
$l=1$	1.8×10^{-2}	3.5×10^{-2}	2.8×10^{-2}	4.2×10^{-3}	7.8×10^{-3}
$l=2$	1.0×10^{-1}	3.8×10^{-2}	4.6×10^{-3}	1.3×10^{-2}	2.3×10^{-3}
$l=3$	4.6×10^{-2}	5.3×10^{-2}	1.0×10^{-2}	9.0×10^{-3}	1.4×10^{-3}

$l=4$	2.0×10^{-4}	7.2×10^{-3}	1.9×10^{-2}	3.6×10^{-3}	3.6×10^{-3}
$l=5$	1.6×10^{-1}	5.3×10^{-2}	2.2×10^{-2}	1.0×10^{-2}	4.9×10^{-3}
$l=6$	1.4×10^{-1}	2.7×10^{-2}	3.8×10^{-3}	1.0×10^{-4}	2.4×10^{-3}
$l=7$	1.2×10^{-1}	9.0×10^{-3}	3.8×10^{-3}	4.1×10^{-3}	8.2×10^{-4}
$l=8$	1.1×10^{-1}	3.3×10^{-2}	7.4×10^{-3}	2.5×10^{-4}	1.0×10^{-4}
$l=9$	1.0×10^{-1}	2.7×10^{-2}	8.3×10^{-3}	2.0×10^{-3}	2.2×10^{-5}
$\|e_m\|_2$	1.39×10^{-1}	5.78×10^{-2}	2.78×10^{-2}	1.37×10^{-2}	6.85×10^{-3}
ρ_m	-	1.26	1.06	1.00	1.00
$Q(m)$	7.51654	7.03992	6.93547	6.91017	6.90390

At the end, suppose that ε is a fixed value in $(0,1)$. For this value of ε, the approximate relations (10) and (25) are exact at the points $t=(i+\varepsilon)h, i=0,1,...,m-1$, and the approximate solutions obtained by this method are more accurate at the points $\{((i+\varepsilon)h,(i+\varepsilon)h), i=0,1,...,m-1\}$ of D, specially when $\varepsilon=\frac{1}{2}$.

8. conclusions

A method based on block-pulse functions and their operational matrix of integration for the solution of linear two-dimensional integral equation is proposed. This approach transforms two-dimensional VIEs of the first and second kind into a family of linear systems of equations. The coefficients matrices of the obtained linear systems, $H^{(\varepsilon)}$ and $I-H^{(\varepsilon)}$, are block lower triangular matrices whose blocks are also lower triangular matrices. This is very satisfactory from the computational point of view, because we need to compute only $\frac{m^2(m+1)^2}{4}$ of integrals in (21) to compute these matrices.

Moreover, we do not use any projection methods such as Galerkin, collocation, etc. to set up the linear systems (46) and (51). So, in comparison with other methods, such as the methods applied in [28, 29], the cost of setting up linear systems is very low. On the other hand, the linear systems (46) and (51) can be solved directly by forward substitution method, which is very advantageous computationally, so that we can increase m to get more accurate approximate solutions without being anxious about the number of operations.

This approach also reduces the two-dimensional FIEs of the first and second kind into linear systems of equations. But for these equations the coefficients matrix of the obtained linear system is a full matrix and for large m solving these systems directly by Gauss elimination method can lead to surprisingly large computing times. But we can seek some ways of reducing this cost, for example constructing an iterative scheme for solving these systems.

The applicability and accuracy of the method is checked on some examples. The validity of arguments of Section 7 can be checked by the numerical results obtained for these examples. For instance, the numerical results reported in the tables show the first order of convergence which is consistent with the claim of Theorem 4.3. Also, all the presented tables suggest us to choose $\varepsilon = \frac{1}{2}$ which is again consistent with the arguments of Section 7.

For Example 6.3, the condition numbers $Q(m)$ which are reported in Table 5 show that the problem is well-conditioned. This confirms the arguments at the end of Section 5. In the case of two-dimensional FIEs of the first kind, because of unexpected oscillations and ill-conditioning of the problem, we do not recommend this method.

Finally, it is clear that the analysis of this paper extends to $N-$dimensional Volterra and Fredholm integral equations. We have dealt with the case $N = 2$ mainly for the sake of clarity.

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