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Formal Local Cohomology Modules and Serre Subcategories

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Abstract

Let $(R,\boldsymbol{m})$ be a Noetherian local ring, $\boldsymbol{a}$ an ideal of R and M a finitely generated R-module. We investigate some properties of formal local cohomology modules with respect to a Serre subcategory. We provide a common language to indicate some properties of formal local cohomology modules.

1. Introduction

Throughout this paper $(R,\boldsymbol{m})$ is a commutative Noetherian local ring, $\boldsymbol{a}$ an ideal of R and M is a finitely generated R- module. For an integer $i \in \mathbb{N}_0, H^i_{\boldsymbol{a}}(N)$ denotes the i-th local cohomology module of M with respect to $\boldsymbol{a}$ as introduced by Grothendieck (cf. [1], [2].

We shall consider the family of local cohomology modules $\{H^i_{\boldsymbol{m}}\left(\frac{M}{a^n M}\right)\}_{n\in\mathbb{N}}$ for a non-negative integer $i \in \mathbb{N}_0$. With natural homomorphisms; this family forms an inverse system. Schenzel introduced the i-th formal local cohomology of M with respect to $\boldsymbol{a}$ in the form of $f^i_{\boldsymbol{a}}(M) := \frac{lim}{n\in\mathbb{N}} H^i_{\boldsymbol{m}}\left(\frac{M}{a^n M}\right)$, which is the i-th cohomology module of the $\boldsymbol{a}$-adic completion of the Čech complex $\check{c}_x \otimes_R M$, where $\underline{x}$ denotes a system of elements of R such that $Rad(\underline{x}, R) = \boldsymbol{m}$ (see [3, Definition 3.1]). He defines the formal grade as $f.grade(\boldsymbol{a}, M) = \inf\{i \in \mathbb{N}_0 \mid f^i_{\boldsymbol{a}}(M) \neq 0\}$. For any ideal $\boldsymbol{a}$ of R and finitely generated R- module M the following statements hold:

(i) (See [3, Theorem 3.11]). If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of finitely generated R- modules, then there is the following long exact sequence:

$$\cdots \to f^i_{\boldsymbol{a}}(M') \to f^i_{\boldsymbol{a}}(M) \to f^i_{\boldsymbol{a}}(M'') \to \cdots .$$

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(ii) (See [3, Theorem 1.3]). $f.grade(\boldsymbol{a}, M) \leq \dim(M) - cd(\boldsymbol{a}, M)$; some properties of formal local cohomology have been presented in [3, 4, 5 and 6].

Throughout this paper $\mathcal{S}$ denotes a Serre subcategory of the category of R- modules and $R-$homomorphisms (we recall that a class $\mathcal{S}$of R-modules is a Serre subcategory of the category of R- modules and R-homomorphisms if $\mathcal{S}$ is closed under taking submodules, quotients and extensions).

Our paper contains three sections. In Section 2, we shall define the formal grade of $\boldsymbol{a}$ with respect to M in $\mathcal{S}$ as the infimum of the integers i such that $f_{\boldsymbol{a}}^i(M) \notin \mathcal{S}$ and is denoted by $f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. (See definition 2.1). Then we shall obtain some properties of this notion. We show that if $\Gamma_{\boldsymbol{a}}(M)$ is a pure submodule of M, then $Hom_R(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))$ and $Hom_R(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t-1}\left(\frac{\mathrm{M}}{\Gamma_{\boldsymbol{a}}(M)}\right))$ belong to $\mathcal{S}$,where $t = f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$.

In Section 3, we shall define the formal cohomological dimension of $\boldsymbol{a}$ with respect to M in $\mathcal{S}$ as the supremum of the integers i such that $f_{\boldsymbol{a}}^i(M) \notin \mathcal{S}$ and is denoted by $f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$. (See definition 3.1). The main result of this section is that if $f_{\boldsymbol{a}}^i(M) \in \mathcal{S}$ and $H_{\boldsymbol{m}}^i(M) \in \mathcal{S}$ for all $i > t$, then $\frac{R}{\boldsymbol{a}} \otimes_R f_{\boldsymbol{a}}^t(M)$ belongs to $\mathcal{S}$.

2. The formal grade of a module in a Serre subcategory

Definition 2.1. The formal grade of $\boldsymbol{a}$ with respect to M in $\mathcal{S}$ is the infimum of the integers i such that $f_{\boldsymbol{a}}^i(M) \notin \mathcal{S}$ and is denoted by $f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$.

Proposition 2.2. Let $(R, \boldsymbol{m})$ be a local ring and $\boldsymbol{a}$ be an ideal of R. If $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated R-modules, then the following statements hold.

(a) $f.grade_{\mathcal{S}}(\boldsymbol{a}, M) \geq \min\{f.grade_{\mathcal{S}}(\boldsymbol{a}, L), f.grade_{\mathcal{S}}(\boldsymbol{a}, N)\}$.

(b) $f.grade_{\mathcal{S}}(\boldsymbol{a}, L) \geq \min\{f.grade_{\mathcal{S}}(\boldsymbol{a}, M), f.grade_{\mathcal{S}}(\boldsymbol{a}, N) + 1\}$.

(c) $f.grade_{\mathcal{S}}(\boldsymbol{a}, N) \geq \min\{f.grade_{\mathcal{S}}(\boldsymbol{a}, L) - 1, f.grade_{\mathcal{S}}(\boldsymbol{a}, M)\}$.

Proof. According to [3, Theorem 3.11], the above short exact sequence induces the following long exact sequence.

$$\cdots \to f_{\boldsymbol{a}}^{i-1}(N) \to f_{\boldsymbol{a}}^i(L) \to f_{\boldsymbol{a}}^i(M) \to f_{\boldsymbol{a}}^i(N) \to f_{\boldsymbol{a}}^{i+1}(L) \to \cdots .$$

So, the result follows.

Corollary 2.3. If $\underline{x} = x_1, \dots, x_n$ is a regular M-sequence, then $f.grade_{\mathcal{S}}\left(\boldsymbol{a}, \frac{M}{\underline{x}M}\right) \geq f.grade_{\mathcal{S}}\,(\boldsymbol{a}, M\,) - n.$

Proof. Consider the following exact sequence ($n \in \mathbb{N}$)

$$0 \to \frac{M}{(x_1, \dots, x_{n-1})M} \xrightarrow{x_n} \frac{M}{(x_1, \dots, x_{n-1})M} \xrightarrow{nat.} \frac{M}{(x_1, \dots, x_n)M} \to 0$$

whenever $n = 1$ by $(x_1, \dots, x_{n-1})M$ we means 0.

Corollary 2.4. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be ideals of R. Then

(a) $f.grade_{\mathcal{S}}\,(\boldsymbol{a} \cap \boldsymbol{b}\,, M\,) \geq$
$\min\{f.grade_{\mathcal{S}}\,(\boldsymbol{a}, M\,), f.grade_{\mathcal{S}}\,(\boldsymbol{b}, M\,), f.grade_{\mathcal{S}}\,((\boldsymbol{a}, \boldsymbol{b}), M) + 1\,)\}.$

(b) $f.grade_{\mathcal{S}}\,((\boldsymbol{a}, \boldsymbol{b}), M\,) \geq$
$\min\{f.grade_{\mathcal{S}}\,(\boldsymbol{a} \cap \boldsymbol{b}\,, M\,) - 1, f.grade_{\mathcal{S}}\,(\boldsymbol{a}, M\,), f.grade_{\mathcal{S}}\,(\boldsymbol{b}, M\,)\}$.

Proof. For all $n \in \mathbb{N}$ there is a short exact sequence as follows:

$$0 \to \frac{M}{\boldsymbol{a}^n M \cap \boldsymbol{b}^n M} \to \frac{M}{\boldsymbol{a}^n M} \oplus \frac{M}{\boldsymbol{b}^n M} \to \frac{M}{(\boldsymbol{a}^n, \boldsymbol{b}^n)M} \to 0.$$

By using [3,Theorem 5.1], the above exact sequence induces the following long exact sequence.

$$\cdots \to \varprojlim_{n\in\mathbb{N}} H^i_{\boldsymbol{m}}\left(\frac{M}{(\boldsymbol{a}\cap\boldsymbol{b})^n M}\right) \to \varprojlim_{n\in\mathbb{N}} H^i_{\boldsymbol{m}}\left(\frac{M}{\boldsymbol{a}^n M}\right) \oplus \varprojlim_{n\in\mathbb{N}} H^i_{\boldsymbol{m}}\left(\frac{M}{\boldsymbol{b}^n M}\right) \to \varprojlim_{n\in\mathbb{N}} H^i_{\boldsymbol{m}}\left(\frac{M}{(\boldsymbol{a},\boldsymbol{b})^n M}\right) \to \cdots.$$

So by using an argument similar to that of Proposition 2.2, the result follows.

Corollary 2.5. Assume that M is a finitely generated R- module and N_1 and N_2 are submodules of M. Then considering the exact sequence $0 \to \frac{M}{N_1 \cap N_2} \to \frac{M}{N_1} \oplus \frac{M}{N_2} \to \frac{M}{N_1 + N_2} \to 0$ we shall have

(a) $f.grade_{\mathcal{S}}\left(\boldsymbol{a}\ ,\ \frac{M}{N_1 \cap N_2}\right) \geq \min\{f.grade_{\mathcal{S}}\left(\boldsymbol{a}\,,\ \frac{M}{N_1}\right), f.grade_{\mathcal{S}}\Big(\boldsymbol{a}\ ,$

MN2 , f.gradeS $\boldsymbol{a}$ *, MN1+ N2 +1}.*

(b) $f.grade_{\mathcal{S}}\left(\boldsymbol{a}\ ,\ \frac{M}{N_1 + N_2}\right) \geq \min\left\{f.grade_{\mathcal{S}}\left(\frac{M}{N_1 \cap N_2}\right) - 1, f.grade_{\mathcal{S}}\Big(\boldsymbol{a}\ ,\right.$

MN1 , f.gradeS $\boldsymbol{a}$ *, MN2 .*

Theorem 2.6. Let $\boldsymbol{a}$ be an ideal of a local ring $(\,R, \boldsymbol{m}\,)$, M be a finitely generated R-module and L be a pure submodule of M. Then $f.grade_{\mathcal{S}}(\boldsymbol{a}\,, L) \geq f.grade_{\mathcal{S}}(\boldsymbol{a}\,, M)$ where $\mathcal{S}$ is a Serre subcategory of the category of R- modules and R-homomorphisms. In particular, $\inf\,\{i | H^i_{\boldsymbol{m}}(L) \notin \mathcal{S}\} \geq \inf\,\{i | H^i_{\boldsymbol{m}}(M) \notin \mathcal{S}\}$.

Proof. Let L be a pure submodule of M. So $\frac{L}{\boldsymbol{a}^n L} \to \frac{M}{\boldsymbol{a}^n M}$ is pure for each $n \in \mathbb{N}$. Now according to [8, Corollary 3.2 (a)] , $H_{\boldsymbol{m}}^i\left(\frac{L}{\boldsymbol{a}^n L}\right) \to H_{\boldsymbol{m}}^i\left(\frac{M}{\boldsymbol{a}^n M}\right)$ is injective. Since inverse limit is a left exact functor, $f_{\boldsymbol{a}}^i(L)$ is isomorphic to a submodule of $f_{\boldsymbol{a}}^i(M)$. Consequently ,$f.grade_{\mathcal{S}}(\boldsymbol{a}, L) \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. If $\boldsymbol{a} = 0$ then, $f.grade_{\mathcal{S}}(0, M) = \inf\{i | H_{\boldsymbol{m}}^i(M) \notin \mathcal{S}\}$ and the result follows.

Corollary 2.7. If $0 \to L \to M \to N \to 0$ is a pure exact sequence of finitely generated R- modules, then $\min\{f.grade_{\mathcal{S}}(\boldsymbol{a}, L),\ f.grade_{\mathcal{S}}(\boldsymbol{a}, N) + 1\} \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$.

Proof. Since L is a pure submodules of M, as a result of the previous theorem, $f.grade_{\mathcal{S}}(\boldsymbol{a}, L) \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. Hence we must prove that $f.grade_{\mathcal{S}}(\boldsymbol{a}, N) + 1 \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. We assume that $i < f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$ and we show that $i < f.grade_{\mathcal{S}}(\boldsymbol{a}, N) + 1$. Consider the following long exact sequence.

$$\cdots \to f_{\boldsymbol{a}}^{i-1}(M) \to f_{\boldsymbol{a}}^{i-1}(N) \to f_{\boldsymbol{a}}^i(L) \to f_{\boldsymbol{a}}^i(M) \to f_{\boldsymbol{a}}^i(N) \to \cdots . \ (**)$$

If $i < f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$, then $f_{\boldsymbol{a}}^0(M), f_{\boldsymbol{a}}^1(M), \dots, f_{\boldsymbol{a}}^{i-1}(M), f_{\boldsymbol{a}}^i(M) \in \mathcal{S}$. On the other hand, since $i < f.grade_{\mathcal{S}}(\boldsymbol{a}, M) \leq f.grade_{\mathcal{S}}(\boldsymbol{a}, L), f_{\boldsymbol{a}}^0(L), \dots, f_{\boldsymbol{a}}^i(L) \in \mathcal{S}$. Hence, it follows from $(**)$ that $f_{\boldsymbol{a}}^0(N), \dots, f_{\boldsymbol{a}}^{i-1}(N) \in \mathcal{S}\ and\ so\ i - 1 < f.grade_{\mathcal{S}}(\boldsymbol{a}, N)$.

Theorem 2.8. Let $(R, \boldsymbol{m})$ be a local ring, $\boldsymbol{a}$ be an ideal of R, $\mathcal{S}$ be a Serre subcategory of the category of R- modules and R_homomorphisms and $M \in \mathcal{S}$ be a finitely generated R- module such that $\Gamma_{\boldsymbol{a}}(M)$ is a pure submodule of M. Then $Hom_R\left(\frac{R}{\boldsymbol{a}}, f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))\right) \in \mathcal{S}$, where $t = f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$.

Proof. Due to the previous theorem, $f.grade_{\mathcal{S}}(\boldsymbol{a}, \Gamma_{\boldsymbol{a}}(M)) \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. If $f.grade_{\mathcal{S}}(\boldsymbol{a}, \Gamma_{\boldsymbol{a}}(M)) > f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$, then the result is obvious. Accordingly, we assume that $f.grade_{\mathcal{S}}(\boldsymbol{a}, \Gamma_{\boldsymbol{a}}(M)) = f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. We know that $Supp(\Gamma_{\boldsymbol{a}}(M)) \subseteq Var(\boldsymbol{a})$. By using [4, Lemma 2.3], $f_{\boldsymbol{a}}^i(\Gamma_{\boldsymbol{a}}(M)) \cong H_{\boldsymbol{m}}^i(\Gamma_{\boldsymbol{a}}(M))$ for all $i \geq 0$. So, if $j < f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$, then $f_{\boldsymbol{a}}^j(\Gamma_{\boldsymbol{a}}(M)) \cong H_{\boldsymbol{m}}^j(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$ and $Ext_R^k(\frac{R}{\boldsymbol{m}}, H_{\boldsymbol{m}}^j(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$ for all $k \geq 0$ and $j < f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. Moreover $Ext_R^t\left(\frac{R}{\boldsymbol{m}}, \Gamma_{\boldsymbol{a}}(M)\right) \in \mathcal{S}$, because $\Gamma_{\boldsymbol{a}}(M) \in \mathcal{S}$. Consequently, according to [7, Theorem 2.2],

$$Hom_R(\frac{R}{\boldsymbol{m}}, H_{\boldsymbol{m}}^t(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}, \text{ where } t = f.grade_{\mathcal{S}}(\boldsymbol{a}, M).$$

Corollary 2.9 With the same notations as Theorem 2.8, let $X \in \mathcal{S}$ be a submodule of $f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))$, where $t = f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$. Then $Hom_R(\frac{R}{\boldsymbol{m}}, \frac{f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))}{X}) \in \mathcal{S}$.

Proof. Consider the long exact sequence:

$$Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))\right) \to Hom_R\left(\frac{R}{\boldsymbol{m}}, \frac{f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))}{X}\right) \to Ext_R^1\left(\frac{R}{\boldsymbol{m}}, X\right). \ (*)$$

In accordance with the previous theorem $Hom_R(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))) \in \mathcal{S}$. Moreover $Ext_R^1(\frac{R}{\boldsymbol{m}}, X) \in \mathcal{S}$. It follows from the exact sequence $(*)$ that $Hom_R\left(\frac{R}{\boldsymbol{m}}, \frac{f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M))}{X}\right) \in \mathcal{S}$.

Theorem 2.10. Suppose that $\boldsymbol{a}$ is an ideal of $(R, \boldsymbol{m})$ and $M \in \mathcal{S}$ is a finitely generated R- module such that $\Gamma_{\boldsymbol{a}}(M)$ is a pure submodule of M. Then $Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right)\right) \in \mathcal{S}$, where $t = f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$.

Proof. One has $f.grade_{\mathcal{S}}(\boldsymbol{a}, \Gamma_{\boldsymbol{a}}(M)) \geq f.grade_{\mathcal{S}}(\boldsymbol{a}, M)$, by Theorem 2.6. Now, the exact sequence $0 \to \Gamma_{\boldsymbol{a}}(M) \to \mathrm{M} \to \frac{M}{\Gamma_{\boldsymbol{a}}(M)} \to 0$ induces the following long exact sequence:

$$\cdots \xrightarrow{\alpha} f_{\boldsymbol{a}}^{t-1}(\Gamma_{\boldsymbol{a}}(M)) \xrightarrow{\beta} f_{\boldsymbol{a}}^{t-1}(M) \xrightarrow{\gamma} f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right) \xrightarrow{\xi} f_{\boldsymbol{a}}^t(\Gamma_{\boldsymbol{a}}(M)) \xrightarrow{\varphi} \cdots. \ (*)$$

Using the exact sequence $(*)$, we obtain the short exact sequence $0 \to \mathrm{Im}(\beta) \to f_{\boldsymbol{a}}^{t-1}(M) \to Im(\gamma) \to 0$. Since $f_{\boldsymbol{a}}^{t-1}(M) \in \mathcal{S}$, $\mathrm{Im}(\beta) \in \mathcal{S}$ and $Im(\gamma) \in \mathcal{S}$. Furthermore, we have the exact sequence $0 \to \mathrm{Im}(\xi) \to H_{\boldsymbol{m}}^t(\Gamma_{\boldsymbol{a}}(M)) \to Im(\varphi) \to 0$ which induces the following long exact sequence:

$$0 \to Hom_R(\frac{R}{\boldsymbol{m}}, \mathrm{Im}(\xi)) \to Hom_R(\frac{R}{\boldsymbol{m}}, H_{\boldsymbol{m}}^t(\Gamma_{\boldsymbol{a}}(M))) \to \cdots.$$

Thus $Hom_R(\frac{R}{\boldsymbol{m}}, \mathrm{Im}(\xi)) \in \mathcal{S}$. Finally, by considering the short exact sequence $0 \to \mathrm{Im}(\gamma) \to f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right) \to Im(\xi) \to 0$ we can conclude that $Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right)\right) \in \mathcal{S}$.

Theorem 2.11. Suppose that R is complete with respect to the $\underline{a}$–adic topology and $M \in \mathcal{S}$ be a finitely generated R- module and t a positive integer such that $f_{\boldsymbol{a}}^i(M) \in \mathcal{S}$ for all $i < t$. Then $Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^t(M)\right) \in \mathcal{S}$.

Proof. We use induction on t. Let t=0. Consider the following isomorphisms.

$$Hom_R(\frac{R}{\underline{m}}, f_{\underline{a}}^0(M)) \cong \varprojlim_{n\in\mathrm{N}} Hom_R(\frac{R}{\underline{m}}, H_{\underline{m}}^0(\frac{M}{\underline{a}^n M})) \cong \varprojlim_{n\in\mathrm{N}} Hom_R(\frac{R}{\underline{m}}, \frac{M}{\underline{a}^n M})$$
$$\cong Hom_R(\frac{R}{\underline{m}}, \varprojlim_{n\in\mathrm{N}} (\frac{M}{\underline{a}^n M})) \cong Hom_R(\frac{R}{\underline{m}}, \hat{M}^{\underline{a}}) \cong Hom_R(\frac{R}{\underline{m}}, M)$$

It is clear that $Hom_R(\frac{R}{\underline{m}}, M) \in S$. So by the above isomorphisms, we deduce that $Hom_R(\frac{R}{\underline{m}}, f_{\underline{a}}^{0}(M)) \in S$.

Suppose that t>0 and the result is true for all integer i less than t. Set N:= $\Gamma_{\mathbf{m}}(M)$. Then $f_{\boldsymbol{a}}^{i}(M) \cong f_{\boldsymbol{a}}^{i}\left(\frac{M}{N}\right)$ $for\ all\ i > 0$, and so we may assume that $depth_R(M) > 0$. There is an $M-regular$ element $x \in \boldsymbol{m}$. The exact sequence $0 \to M \xrightarrow{x} M \to \frac{M}{xM} \to 0$ induces the following long exact sequence:

$$\cdots \to f_{\boldsymbol{a}}^{t-2}(M) \xrightarrow{x} f_{\boldsymbol{a}}^{t-2}(M) \xrightarrow{f} f_{\boldsymbol{a}}^{t-2}\left(\frac{M}{xM}\right)$$

$$\to f_{\boldsymbol{a}}^{t-1}(M) \xrightarrow{x} f_{\boldsymbol{a}}^{t-1}(M) \xrightarrow{g} f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{xM}\right)$$

$$\to f_{\boldsymbol{a}}^{t}(M) \xrightarrow{x} f_{\boldsymbol{a}}^{t}(M) \xrightarrow{h} \cdots. \quad (*)$$

Using the exact sequence $(*)$ we obtain the short exact sequence

$$0 \to \frac{f_{\boldsymbol{a}}^{t-1}(M)}{xf_{\boldsymbol{a}}^{t-1}(M)} \to f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{xM}\right) \to (0 \underset{f_{a}^{t}(M)}{:} x) \to 0.$$

Now, this exact sequence induces the following long exact sequence:

$$0 \to Hom_R\left(\frac{R}{\boldsymbol{m}}, \frac{f_{\boldsymbol{a}}^{t-1}(M)}{xf_{\boldsymbol{a}}^{t-1}(M)}\right) \to Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{xM}\right)\right) \to Hom_R\left(\frac{R}{\boldsymbol{m}}, (0 \underset{f_{a}^{t}(M)}{:} x)\right) \to$$
$$Ext_R^1\left(\frac{R}{\boldsymbol{m}}, \frac{f_{\boldsymbol{a}}^{t-1}(M)}{xf_{\boldsymbol{a}}^{t-1}(M)}\right) \to \cdots . (**)$$

By using $(*)$, $f_{\boldsymbol{a}}^{i}\left(\frac{M}{xM}\right) \in \mathcal{S}$ for all $i < t-1$. Therefore by the induction hypothesis $Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t-1}\left(\frac{M}{xM}\right)\right) \in \mathcal{S}$. Furthermore $Ext_R^1\left(\frac{R}{\boldsymbol{m}}, \frac{f_{\boldsymbol{a}}^{t-1}(M)}{xf_{\boldsymbol{a}}^{t-1}(M)}\right) \in \mathcal{S}$ because $f_{\boldsymbol{a}}^{t-1}(M) \in \mathcal{S}$. Thus in accordance with $(**)$, $Hom_R\left(\frac{R}{\boldsymbol{m}}, (0 \underset{f_{a}^{t}(M)}{:} x)\right) \in \mathcal{S}$. Since $x \in \boldsymbol{m}$ according to [9,10.86] we have the following isomorphisms.

$$Hom_R\left(\frac{R}{\boldsymbol{m}}, (0 \underset{f_{a}^{t}(M)}{:} x)\right) \cong Hom_R\left(\frac{R}{\boldsymbol{m}}, Hom_R\left(\frac{R}{xR}, f_{\boldsymbol{a}}^{t}(M)\right)\right) \cong$$
$$Hom_R\left(\frac{R}{\boldsymbol{m}} \otimes_R \frac{R}{xR}, f_{\boldsymbol{a}}^{t}(M)\right) \cong Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t}(M)\right).$$

Consequently $Hom_R\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{t}(M)\right) \in \mathcal{S}$.

3. The formal cohomological dimension in a Serre subcategory

We recall from [3,Theorem 1.1] that for a finitely generated R- module M, $\sup\{i \in \mathbb{N}_0 | f_{\boldsymbol{a}}^i(M) \neq 0\} = \dim(\frac{M}{\boldsymbol{a}M})$.

Definition 3.1. The formal cohomological dimension of M with respect to a in $\mathcal{S}$ is The supremum of the integers i such that $f_{\boldsymbol{a}}^i(M) \notin \mathcal{S}$ and is denoted by $f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$.

Theorem 3.2. Suppose that $\mathcal{S}$ is a Serre subcategory of the category of R- modules and $R-$ homomorphisms and L and N are two finitely generated R- modules such that $Supp_R(L) \subseteq Supp_R(N)$. Then $f.cd_{\mathcal{S}}(\boldsymbol{a}, L) \leq f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$.

Proof. It is enough to prove that $f_{\boldsymbol{a}}^i(L) \in \mathcal{S}$ for all $i > f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$ and all finitely generated R- module L such that $Supp_R(L) \subseteq Supp_R(N)$. We use descending induction on i.For all i $>$ dim $(\frac{L}{\boldsymbol{a}L})$+ $f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$, $f_{\boldsymbol{a}}^i(L) = 0 \in \mathcal{S}$. Let $i > f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$ and the result is proved for $i+1$. By Gruson's theorem, there is a chain $0 = L_0 \subset L_1 \subset \cdots \subset L_l = L$ of submodules of L such that $\frac{L_i}{L_{i-1}}$ is a homomorfic image of a direct sum of finitely many copies of N. Consider the exact sequence $0 \to L_{i-1} \to L_i \to \frac{L_i}{L_{i-1}} \to 0$ $(i = 0,1,\dots,l)$. We may assume that $l = 1$. The exact sequence $0 \to K \to \oplus_{j=1}^t N \to L \to 0$ where K is a finitely generated R- module iduces the following long exact sequence:

$$\cdots \to f_{\boldsymbol{a}}^i\left(\oplus_{j=1}^t N\right) \to f_{\boldsymbol{a}}^i(L) \to f_{\boldsymbol{a}}^{i+1}(K) \to \cdots. (*)$$

Based on the induction hypothesis $f_{\boldsymbol{a}}^{i+1}(K) \in \mathcal{S}$. Moreover $f_{\boldsymbol{a}}^i\left(\oplus_{j=1}^t N\right) = \oplus_{j=1}^t f_{\boldsymbol{a}}^i(N) \in \mathcal{S}$ for all $i > f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$. Hence it follows from the exact sequence $(*)$ that $f_{\boldsymbol{a}}^i(L) \in \mathcal{S}$.

The next example shows that even if $Supp_R(M) = Supp_R(N)$, then it may not true that $f.grade_{\mathcal{S}}(\boldsymbol{a}, M) = f.grade_{\mathcal{S}}(\boldsymbol{a}, N)$.

Example 3.3. (See [4, Example 4.3 (i)]) Let $(R, \boldsymbol{m})$ be a 2 dimensional complete regular local ring, $\mathcal{S} = 0$ and $\boldsymbol{a}$ be an ideal of R with $\dim\left(\frac{R}{\boldsymbol{a}}\right) = 1$. Then by using [5,Theorem 1.1], $f.grade_{\mathcal{S}}(\boldsymbol{a}, R) = 1$ and $f.grade_{\mathcal{S}}\left(\boldsymbol{a}, \frac{R}{\boldsymbol{m}}\right) = 0$. Set $M := R \oplus \frac{R}{\boldsymbol{m}}$. Then $Supp_R(M) = Supp_R(R)$. But

$$f.grade_{\mathcal{S}}(\boldsymbol{a}, M) = \inf\left\{f.grade_{\mathcal{S}}(\boldsymbol{a}, R), f.grade_{\mathcal{S}}\left(\boldsymbol{a}, \frac{R}{\boldsymbol{m}}\right)\right\} = 0.$$

Corollary3.4. For all $x \in \boldsymbol{a}$, $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) \geq f.cd_{\mathcal{S}}\left(\boldsymbol{a}, \frac{M}{xM}\right)$.

Corollary3.5. Suppose that $0 \to L \to M \to N \to 0$ is an exact sequence of finitely generated R- modules. Then $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) = \max\{f.cd_{\mathcal{S}}(\boldsymbol{a}, L), f.cd_{\mathcal{S}}(\boldsymbol{a}, N)\}$.

Proof. Since $Supp_R(M) = Supp_R(L) \cup Supp_R(N)$ by referring to Theorem 3.2 we deduce that $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) \geq f.cd_{\mathcal{S}}(\boldsymbol{a}, L)$ and $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) \geq f.cd_{\mathcal{S}}(\boldsymbol{a}, N)$. Therefore $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) \geq max\,\{f.cd_{\mathcal{S}}(\boldsymbol{a}, L), f.cd_{\mathcal{S}}(\boldsymbol{a}, N)\}$.

Next we prove that $max\,\{f.cd_{\mathcal{S}}(\boldsymbol{a}, L), f.cd_{\mathcal{S}}(\boldsymbol{a}, N)\} \geq f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$.

Let $i > max\,\{f.cd_{\mathcal{S}}(\boldsymbol{a}, L), f.cd_{\mathcal{S}}(\boldsymbol{a}, N)\}$. Then $f_{\boldsymbol{a}}^i(N), f_{\boldsymbol{a}}^i(L) \in \mathcal{S}$ and from the exact sequence $f_{\boldsymbol{a}}^i(L) \to f_{\boldsymbol{a}}^i(M) \to f_{\boldsymbol{a}}^i(N)$ we conclude that $f_{\boldsymbol{a}}^i(M) \in \mathcal{S}$. Thus, $max\{f.cd_{\mathcal{S}}(\boldsymbol{a}, L), f.cd_{\mathcal{S}}(\boldsymbol{a}, N)\} \geq f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$.

We recall that the cohomological dimension of an R- module M with respect to an ideal $\boldsymbol{a}$ of R in $\mathcal{S}$ is defind as

$$cd_{\mathcal{S}}(\boldsymbol{a}, M) := \sup\{i \in \mathbb{N}_0 | H_{\boldsymbol{a}}^i(M) \notin \mathcal{S}\}.$$

The following lemma shows that when we considering the Artinianness of $f_{\boldsymbol{a}}^i(M)$, we can assume that M is $\boldsymbol{a}$-$torsion$-$free$.

Lemma 3.6. Suppose that $\boldsymbol{a}$ is an ideal of a local ring $(R, \boldsymbol{m})$ and t be a non-negative integer. If $H_{\boldsymbol{m}}^{\,i}(M) \in \mathcal{S}$ for all $i \geq t$, then the following are equivalent:

(a) $f_{\boldsymbol{a}}^i(M) \in \mathcal{S}$ for all $i \geq t$.

(b) $f_{\boldsymbol{a}}^i\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right) \in \mathcal{S}$ for all $i \geq t$.

Proof. According to the hypothesis $t > cd_{\mathcal{S}}(\boldsymbol{m}, M)$. On the other hand $Supp_R(\Gamma_{\boldsymbol{a}}(M)) \subseteq Supp_R(M)$. So by referring to [7,Theorem 3.5], $cd_{\mathcal{S}}(\boldsymbol{m}, \Gamma_{\boldsymbol{a}}(M)) \leq cd_{\mathcal{S}}(\boldsymbol{m}, M)$. Thus, $t > cd_{\mathcal{S}}(\boldsymbol{m}, \Gamma_{\boldsymbol{a}}(M))$ and $H_{\boldsymbol{m}}^i(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$ for all $i \geq t$. Now, consider the following long exact sequence:

$$\cdots \to f_{\boldsymbol{a}}^i(\Gamma_{\boldsymbol{a}}(M)) \to f_{\boldsymbol{a}}^i(M) \to f_{\boldsymbol{a}}^i\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right) \to f_{\boldsymbol{a}}^{i+1}(\Gamma_{\boldsymbol{a}}(M)) \to \cdots . (*)$$

According to [4,Lemma 2.3] $f_{\boldsymbol{a}}^i(\Gamma_{\boldsymbol{a}}(M)) \cong H_{\boldsymbol{m}}^i(\Gamma_{\boldsymbol{a}}(M))$. By using the hypothesis $f_{\boldsymbol{a}}^i(\Gamma_{\boldsymbol{a}}(M)) \in \mathcal{S}$ for all $i \geq t$. So it follows from the exact sequence $(*)$ that $f_{\boldsymbol{a}}^i(M) \in \mathcal{S}$ if and only if $f_{\boldsymbol{a}}^i\left(\frac{M}{\Gamma_{\boldsymbol{a}}(M)}\right) \in \mathcal{S}$ for all $i \geq t$.

Theorem 3.7. Let $(R, \boldsymbol{m})$ be a local ring and $M \in \mathcal{S}$ be a finitely generated R- module of dimension d such that $cd_{\mathcal{S}}(\boldsymbol{m}, M) \leq f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$. Then $\frac{f_{\boldsymbol{a}}^t(M)}{\boldsymbol{a} f_{\boldsymbol{a}}^t(M)} \in \mathcal{S}$ where $t = f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$.

Proof. We use induction on $d = \dim(M)$. If $d = 0$, then $\dim\left(\frac{M}{\boldsymbol{a}M}\right) = 0$. Accordingly to [3, Theorem 1.1], $f_{\boldsymbol{a}}^i(M) = 0$ for all $i > 0$.

Moreover $f_{\boldsymbol{a}}^{0}(M) \cong M \in \mathcal{S}$. By definition $H_{\boldsymbol{m}}^{i}(M) \in \mathcal{S}$ for all $i > t$. Therefore from the above lemma we can assume that M is $\boldsymbol{a}$-$torsion$-$free$ and there is an M -$regular$ element $x \in \boldsymbol{a}$. Consider the long exact sequence :

$$\cdots \to f_{\boldsymbol{a}}^{i}(M) \xrightarrow{x} f_{\boldsymbol{a}}^{i}(M) \xrightarrow{f} f_{\boldsymbol{a}}^{i}\left(\frac{M}{xM}\right) \xrightarrow{g} f_{\boldsymbol{a}}^{i+1}(M) \xrightarrow{h} \cdots . (*)$$

By using the hypothesis $f_{\boldsymbol{a}}^{i}(M) \in \mathcal{S}$ for all $i > t$ (because $t = f.cd_{\mathcal{S}}(\boldsymbol{a}, M)$). So using the above long exact sequence $f_{\boldsymbol{a}}^{i}\left(\frac{M}{xM}\right) \in \mathcal{S}$ for all $i > t$. By induction hypothesis, $\frac{f_{\boldsymbol{a}}^{t}\left(\frac{M}{xM}\right)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}\left(\frac{M}{xM}\right)} \in \mathcal{S}$ because $\dim\left(\frac{M}{xM}\right) = \dim(M) - 1$.

Afterwards from the exact sequence $(*)$ we get the following short exact sequence.

$$0 \to Im(f) \to f_{\boldsymbol{a}}^{t}\left(\frac{M}{xM}\right) \to Im(g) \to 0$$

So we obtain the following long exact sequence.

$$\ldots \to Tor_{1}^{R}\left(\frac{R}{\boldsymbol{a}}, Im(g)\right) \to \frac{Im(f)}{\boldsymbol{a}Im(f)} \to \frac{f_{\boldsymbol{a}}^{t}\left(\frac{M}{xM}\right)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}\left(\frac{M}{xM}\right)} \to \frac{Im(g)}{\boldsymbol{a}Im(g)} \to 0.$$

Since $f_{\boldsymbol{a}}^{t}(M) \in \mathcal{S}$ and $Im(g)$ is a submodule of $f_{\boldsymbol{a}}^{t+1}(M)$, we deduce that $Tor_{1}^{R}(\frac{R}{\boldsymbol{a}}, Im(g)) \in \mathcal{S}$. On the other hand, $\frac{f_{\boldsymbol{a}}^{t}\left(\frac{M}{xM}\right)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}\left(\frac{M}{xM}\right)} \in \mathcal{S}$. Therefore, $\frac{Im(f)}{\boldsymbol{a}Im(f)} \in \mathcal{S}$ by the above long exact sequence.

Now, consider the following long exact sequence.

$$\frac{f_{\boldsymbol{a}}^{t}(M)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}(M)} \xrightarrow{x} \frac{f_{\boldsymbol{a}}^{t}(M)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}(M)} \to \frac{Im(f)}{\boldsymbol{a}Im(f)} \to 0.$$

So, $\frac{f_{\boldsymbol{a}}^{t}(M)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}(M)} \cong \frac{Im(f)}{\boldsymbol{a}Im(f)}$ because $x \in \boldsymbol{a}$. Consequently, $\frac{f_{\boldsymbol{a}}^{t}(M)}{\boldsymbol{a}f_{\boldsymbol{a}}^{t}(M)} \in \mathcal{S}$.

Proposition 3.8. For a finitely generated R- module M,

$$f.cd_{\mathcal{S}}(\boldsymbol{a}, M) = max\,\{f.cd_{\mathcal{S}}\left(\boldsymbol{a}, \frac{R}{P}\right) | P \in Ass_{R}(M)\}.$$

Proof. Set $N := \underset{P \in Ass_{R}(M)}{\oplus} \frac{R}{P}$. Then $Supp_{R}(M) = Supp_{R}(N)$. So, by Theorem 3.2 and Corollary 3.5, $f.cd_{\mathcal{S}}(\boldsymbol{a}, M) = f.cd_{\mathcal{S}}(\boldsymbol{a}, N) = max\,\{f.cd_{\mathcal{S}}\left(\boldsymbol{a}, \frac{R}{P}\right) | P \in Ass_{R}(M)\}$.

Proposition 3.9. Assume that $\boldsymbol{a}$ is an ideal of the local ring $(R, \boldsymbol{m})$. Then $Hom_{R}(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{0}(M)) \in \mathcal{S}$ if and only if. $Hom_{R}(\frac{R}{\boldsymbol{m}}, \widehat{M}^{\boldsymbol{a}}) \in \mathcal{S}$.

Proof. It is enough to consider the following isomorphisms

$$Hom_{R}\left(\frac{R}{\boldsymbol{m}}, f_{\boldsymbol{a}}^{0}(M)\right) \cong \underset{n \in \mathbb{N}}{\underleftarrow{lim}}\, Hom_{R}\left(\frac{R}{\boldsymbol{m}}, H_{\boldsymbol{m}}^{0}\left(\frac{M}{\boldsymbol{a}^{n}M}\right)\right) \cong \underset{n \in \mathbb{N}}{\underleftarrow{lim}}\, Hom_{R}\left(\frac{R}{\boldsymbol{m}}, \frac{M}{\boldsymbol{a}^{n}M}\right)$$
$$\cong Hom_{R}\left(\frac{R}{\boldsymbol{m}}, \underset{n \in \mathbb{N}}{\underleftarrow{lim}}\, \frac{M}{\boldsymbol{a}^{n}M}\right) \cong Hom_{R}\left(\frac{R}{\boldsymbol{m}}, \widehat{M}^{\boldsymbol{a}}\right).$$

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References

1. M. P. Brodmann, R. Y. Sharp, "Local cohomology: An Algebraic introduction with geometric application", Cambridge University Press (1998).
2. A. Grothendick, "Local cohomology, Notes by R.Hartshorne", lect. Notes in Math., 20, Springer (1966).
3. P. Schenzel, "On formal local cohomology and connectedness", J. Algebra, 315 (2007) 894-923.
4. M.Asgharzadeh, K. Divaani-Aazar, "Finiteness properties of formal local cohomology modules and Cohen-Macaulayness", Comm. in Algebra, 39, (2011) 1082-1103.
5. M. Eghbali, "On Artinianness of formal local cohomology, colocalization and coassociated primes", Journal Math. Scand (2012) to appear.
6. Yan Gu, "The Artinianness of formal local cohomology modules", to appear in Bull. Malay. Math. Soc.
7. M. Asgharzadeh, M. Tousi, "A unified approach to local cohomology modules using serre classes", Canadian Math. Bull. , Vol.53, no.1 (2010)1-10.
8. G. Kempf, "The Hochster-Roberts Theorem of invariant theory", Michigan Math. J.26 (1), (1979) 19-32.
9. J. J. Rotman, "An Introduction to homological algebra", Pure and Applied Mathematics 85. Academic Press, Inc., New York (1979).