

# Quasi- Secondary Submodules

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## Abstract

Let  $R$  be a commutative ring with non-zero identity and  $M$  be a unitary  $R$ -module. Then the concept of quasi-secondary submodules of  $M$  is introduced and some results concerning this class of submodules is obtained.

## 1. Introduction

Throughout this paper all rings are commutative with non-zero identity and all modules are unitary. In [4] L.Fuchs introduced and studied the concept of quasi-primary ideals (see also [5]). An ideal  $I$  of a ring  $R$  is called a *quasi-primary* ideal of  $R$  if the radical of  $I$  is a prime ideal of  $R$ . This concept then generalized to modules, i.e., the concept of quasi-primary submodules of a module introduced and developed in [3]. Here, we introduce the dual notation, that is, the quasi-secondary submodules of a module and obtain some results concerning this class of submodules. In section 2, we obtain some preliminary properties of quasi-secondary submodules. Section 3 is devoted to the quasi-secondary submodules of a multiplication module. Now we define some concepts which will be needed in sequel.

Let  $M$  be an  $R$ -module and  $N$  a submodule of it. The ideal  $\{r \in R \mid rM \subseteq N\}$  will be denoted by  $(N \dot{\vdash}_R M)$ ; in particular  $(0 \dot{\vdash}_R M)$  is called the annihilator of  $M$ . A non-zero submodule  $N$  of  $M$  is called a *secondary* (resp. *second*) submodule of  $M$  if for each  $r \in R$  the homothety  $N \xrightarrow{r} N$  is surjective or nilpotent (resp. surjective or zero). In this case  $\sqrt{(0 \dot{\vdash}_R N)}$  is a prime ideal, say  $p$ , and we call  $N$  a *p-secondary* (resp. a *p-second*) submodule of  $M$ . We refer readers for more details concerning secondary (resp. second) submodule to [9] (resp. [12]).

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An  $R$ -module  $M$  is said to be a *multiplication* module if for each submodule  $N$  of  $M$  there exists an ideal  $I$  of  $R$  such that  $N = IM$ . It is easy to see that in this case  $N = (N_R M)M$ . Also the ideal  $\theta(M)$  is defined as  $\theta(M) := \sum_{m \in M} (Rm_R M)$ . If  $M$  is a multiplication module and  $N$  is a submodule of it, then  $M = \theta(M)M$  and  $N = \theta(M)N$ . (see [1]). An  $R$ -module  $M$  is *sum-irreducible* if  $M \neq 0$  and the sum of any two proper submodules of  $M$  is always a proper submodule. Finally a proper submodule  $N$  of an  $R$ -module  $M$  is called a *prime submodule* if for each  $r \in R$  the homothety  $M/N \xrightarrow{r} M/N$  is either injective or zero. This implies that  $\text{Ann}(M/N) = p$  is a prime ideal of  $R$ , and  $N$  is said to be a  *$p$ -prime submodule* (c.f. [7], [8], [10] and [11]).

## 2. Quasi-Secondary Submodules

The starting point of this section is the definition of quasi-secondary submodules of a module.

**Definition 2.1.** Let  $M$  be a non-zero  $R$ -module. Then the non-zero submodule  $N$  of  $M$  is said to be *quasi-secondary* if  $\sqrt{(0_R N)} = p$  where  $p$  is a prime ideal of  $R$ . It is obvious that every secondary (or second) submodule of a module is a quasi-secondary submodule, but the converse is not true in general. For example,  $2\mathbb{Z}$  is a 0-quasi-secondary submodule of the  $\mathbb{Z}$ -module  $\mathbb{Z}$  but it is not 0-secondary (or 0-second) submodule. (Here  $\mathbb{Z}$  denotes the set of all integers.)

### Remark 2.2.

- (i) Let  $M$  be a non-zero  $R$ -module and  $N$  a submodule of it such that  $\sqrt{(0_R N)} = m(m \in \text{Max}(R))$ . Then  $N$  is  *$m$ -secondary* ( *$m$ -second*).
- (ii) Every quasi-secondary submodule of a module over a zero-dimensional ring (i.e., a ring in which every prime ideal is a maximal ideal) is secondary.
- (iii) Every quasi-secondary submodule of a module over a D.V.R is secondary.

**Definition 2.3.** Let  $M$  be an  $R$ -module and  $N$  a submodule of  $M$ . An element  $r$  of  $R$  is called *co-primal* to  $N$  if  $rN = N$ . Denote by  $W(N)$  the set of all elements of  $R$  that are not co-primal to  $N$ . The submodule  $N$  is said to be a *co-primal submodule* of  $M$  if  $W(N)$  is an ideal of  $R$ . This ideal is always a prime ideal. In this case we say that  $N$  is a  *$p$ -co-primal* submodule of  $M$ . The class of co-primal submodules of a module is a

fairly large class. For example , all secondary (second) submodules are co-primal. Also it is easy to see that a sum-irreducible submodule of a module is co-primal. But, in general, a quasi-secondary submodule of a module may not be a co-primal submodule. (consider the  $Z$ -module  $Z$ .) It is worth to mention that in [2] the term secondal is used for co-primal submodules. The next proposition characterizes those  $p$ -quasi- secondary submodules which are  $p$ -co-primal.

Proposition 2.4. Let  $N$  be a  $p$ -quasi-secondary submodule of an  $R$ -module  $M$ . Then  $N$  is a  $p$ -co-primal submodule of  $M$  if and only if it is a  $p$ -secondary submodule of  $M$ .

Proof  $\Rightarrow$ ) Let  $N \xrightarrow{r} N$  be the  $R$ -endomorphism of  $N$  given by multiplication by  $r$  of  $R$  and  $rN \neq N$ . Then by our assumption  $r \in p = \{s \in R \mid sN \neq N\}$ . On the other hand,  $p = \sqrt{0}_R N$  and so there exists a positive integer  $t$  such that  $r^t N = 0$ . The result follows.  $\Leftarrow$ ) Is obvious.

The proof of two next propositions is easy and so we state them without proof.

Proposition 2.5. Let  $M$  be a module over an integral domain and  $N$  be a  $0$ - co-primal submodule of  $M$ . Then  $N$  is  $0$ -secondary.

Proposition 2.6. Let  $M$  be an  $R$ -module and  $N_1, N_2, \dots, N_t$  be submodules of  $M$ . Then

- (i) Suppose that for  $i = 1, 2, N_i$  is  $p_i$ -quasi-secondary. Then  $N_1 + N_2$  is quasi-secondary if and only if  $p_1 \subseteq p_2$  or  $p_2 \subseteq p_1$
- (ii) If  $N_1, \dots, N_t$  are  $p$ -quasi-secondary, then  $N_1 + \dots + N_t$  is a  $p$ -quasi-secondary submodule of  $M$ .
- (iii) If  $N_1 + \dots + N_2$  is a  $p$ -quasi-secondary submodule of  $M$ . Then  $N_j$  is  $p$ -quasi-secondary for some  $j, 1 \leq j \leq t$ .

### 3. Multiplication Modules

In this short section we give a property of quasi-secondary submodules of a multiplication module .

**Lemma 3.1.** let  $M$  be a multiplication module and  $N$  be a  $p$ -quasi-secondary submodule of  $M$ . Then  $\theta(M) \not\subseteq p$ .

**Proof.** Suppose that  $\theta(M) \subseteq p$  and  $0 \neq n \in N$ . Then  $Rn = \theta(M)Rn \subseteq pn$ . Hence  $n = p_0 n$  for some  $p_0 \in p$ . By our assumption there exists a positive integer  $t$  such that  $p_0^t N = 0$ . Therefore  $n = p_0^t n = 0$ , a contradiction.

**Theorem 3.2.** Suppose that  $M$  is a faithful multiplication module and  $N$  a  $p$ -quasi-secondary submodule of  $M$ . Then  $pM$  is a prime submodule of  $M$ . In particular, if  $p \in \max(R)$ , then  $pM$  is a maximal submodule of  $M$ .

**Proof.** By Lemma 3.1,  $\theta(M) \not\subseteq p$ . Now suppose that  $pM = M = RM$ . Then by [1, Theorem 1.5]  $R \cap \theta(M) = \theta(M) = p \cap \theta(M)$  and hence  $\theta(M) \subseteq p$  which is a contradiction. Thus  $pM \neq M$  and the result of the first part follows from [6, Lemma 2.4(2)]. The last part can be deduced from the first part and [6, Corollary 2.7]

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