Some Aspects of Hypergroup Algebras

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Abstract

For a locally compact Hausdorff space X, M(X), the space of bounded regular Borel measures on X, is a Banach space which can be made into a Bananch algerbra by defining the convolution $(\mu,\nu) \longrightarrow \mu * \nu$ *by* $\mu * \nu = \iint \lambda_{(x,y)} d\mu(x)\, d\nu(y)$ *where* $\lambda_{(x,y)}$ *is a probability measure. We define L(X) to be the set of all measures* μ *on X for wich the function* $x \longrightarrow |\mu| * \delta_x$ *is weakly continuous. We shall study some aspects of L(X).*[1]

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1. introduction.

The purpose of this paper is to study some aspects of hypergroups. The idea of convolution operator on a locally compat space goes back a log time ago, and the historical points can be found in Ross [10].

In early seventies three mathematicians have been concerned with this theory. Dunkl [3], and spector [2] use the term hypergroup for their systems while Jewett [6] calls them convos. We shall adopt Dunkl's axioms.

Throughout X will be a locally compact Hausdorff space. We reacll some standard notations of [3]. The spaces $C_c(X)$, $C_b(X)$ and $C_o(X)$ will denote the space of continuous complex valued functions with compact support, the space of continuous complex bounded functions and the space of complex continuous functions vanishing at infinity on X each of them with uniform norm. The dual space of $C_o(X)$ is just $M(X)$, the space of finite regular Borel measures on X and $M_p(X)$ will denote the set of positive probability measures on X, i. e.

$\{\mu \mid \mu \in M(X), \mu(X) = 1\}$. The value of a functional μ at an elemet f will be denoted by $\mu(f)$ and thus $\mu(f) = \int f d\mu$. We use the familiar symbol δ_x for the special functional $f \longrightarrow f(x)$. The support of a measure μ will be denoted by $supp\mu$.

On $M(X)$ we define the convolution by $\mu \star \nu = \iint \lambda_{(x,y)} d\mu(X) d\nu(y)$, where $\lambda_{(x,y)}$ is a probabilitly measure on X and $\lambda_{(x,y)} = \lambda_{(x,y)}$. With this definition $M(X)$ becomes a convoltuion Bananch algebra and X is called a commutative hypergroup. Our defintion is based on [3] and [8]

Propostion 1.1.

With the addtion, the above convolution an measure norm, $M(X)$ is a commutative Banac algebra. Mereover, $M_p(X) \star M_p(X) \subseteq M_p(X)$ [3].

The last part of this propostition states tha $M_p(X)$ is a semigroup. We note that for $x,y \in X$ w have δ_x, $\delta_y \in M_p(X)$, and write $\lambda_{(x,y)} = \delta_x \star \delta_y$ ∈ $M_p(X)$ [3].

definition 1.2.

For $\psi \in C_c(X)$, $x \in X$, $\mu \in M(X)$ define th function $x \longrightarrow R(x)\psi$ by $R(x)\psi(y) = \lambda_{(x,y)}(\psi)$ an the function $\mu \longrightarrow R(\mu)\psi$ by

$$R(\mu)\psi(y) = \int R(z)\psi(y) d\mu(z) \quad (y \in X)$$

Lemma 1.4.

If $(\mu_\alpha)_\alpha$ is a net such that $\mu_\alpha \longrightarrow \mu$ in th weak*-topology , then $\mu_\alpha \star \nu \longrightarrow \mu \star \nu$ in th weak*-topology for all $\mu \in M(X)$.

The proof of the above lemma is a direc consequence of the foregoing proposition an therefore omitted.

Our main interest is to isolate a subalgebra o $M(X)$ which reduces to $L^1(G)$ when $X=G$, is locally compact group.

Definition 1.5.

We define $L(X)$ to be the set of all measure $\mu \in M(X)$ such that the function $x \longrightarrow |\mu| \star \delta_x$ i weakly continous from X into $M(X)$.

6.

ot only a subalgebra of $M(X)$ but also

$M(X)$.

$$S = \overline{\bigcup_{\mu\in L(X)} supp\ \mu}$$

1.7.

ospace S has a hypergroup structure.

$\in S$, and Let $(U_\alpha),(V_\beta)$ be two nets of ighbourhoods of x,y respectively. Then e nets of positive measures (μ_α), (ν_β) in ach that $supp\mu_\alpha\in U_\alpha$, $supp\nu_\beta\in V_\beta$ and $= \|\mu_\beta\| =1$. Clearly $\lambda_{(x,y)}$ is equal to the mit of $(\mu_\alpha * \nu_\beta)$. Since μ_α, $\nu_\beta\in L(X)$, $\in L(X)$ and therefore $suppu\lambda_{(x,y)}\in S$. Thus $_{)}(t) = 1$. Now, let $\psi\in C_c(S)$ we must show e function $(x,y)\longrightarrow \int_S \psi d\lambda_{(x,y)}$ is in $C_b(S\times S)$ $\longrightarrow \int_S \psi d\lambda_{(x,y)}$ is in $C_c(S)$ for every $y\in S$. plying the Tietze extession theorem to the oint compactification of X, we can find an ion ψ_1 of ψ to X with $\psi\in C_c(X)$. Hence $_{(x,y)} = \lambda_{(x,y)}(\psi_1)$ and thus $(x,y) \longrightarrow$ $_{(x,y)}$ is in $C_b(S\times S)$ for every $y\in S$. This letes the proof.

general S is not the whole space X, however now on we restrict our attention to the ergroup" S or in orther words we assume that X.

Thorem 1.8.

There exists a net in $L(X)$ wich is positive and is a weak approximate identity of norm 1 for $L(X)$.

Proof.

Let U be a compact neighbourhood of e. Let A be the collection of all compact neighbourhoods of e contained in U which forms a directed set with set inclusion order. Let $(\nu_\alpha)_\alpha$ be a net of positive normalized measures in $L(X)$ which is supported in $(U_\alpha)_\alpha$ and thus $\nu_\alpha(X\backslash\alpha) = 0$. Let $h\in M(X)$ and $\varepsilon>0$. Since $x \longrightarrow |\mu| * \delta_x$ is weak-continuous $(\mu\in L(X))$, $N = \{x \,\|\, h(\mu*\delta_x) - h(\mu)|<\varepsilon\}$ is a neighbourhood of e. So there is an $\alpha_o\in A$ such that $U_{\alpha_o}\in N$. Let $\alpha>\alpha_0$. Then

$$| h(\mu*\nu_\alpha) - h(\mu) | =$$

$$|\int h(\mu*\delta_\alpha)d\nu_\alpha(x) - \int h(\mu)d\nu_\alpha(x)|$$

$$\leq \int U_\alpha |h(\mu*\delta_\alpha) - h(\mu)|d\nu_\alpha(x)<\varepsilon$$

Theorem 1.9.

There exists a net in $L(X)$ which is a positive approximate identity of norm 1 for $L(X)$.

Proof.

The result is a direct cosequence of the last theorem and ([2], Chapter 10).

The most important property of $L(X)$ is the norm continiuty of $x \longrightarrow |\mu| * \delta_x(\psi)$ into $L(X)$. For this we need some elementary and also basic results.

Lemma 1.10.

Let $\psi \in C_o(X)$ and $\mu \in M(X)$. Then the function $x \longrightarrow \mid \mu \mid \star \delta_x(\psi)$ is continuous on X [3].

Propositon 1.11.

Let τ be a topology on $M(X)$ finer than the weak*-topology. Then the follwing are equivalent.

(i) The function $x \longrightarrow |\mu| \star \delta_x$ is τ-continuous.

(ii) $A = \{|\mu| \star \delta_x \in K\}$ is τ-compact for each compact set $K \subseteq X$.

Proof.

Clearly (i) implies (ii). Now let (ii) hold. Since the τ-topology is finer than the weak-topolgy and A is τ-compact, the topologies coincide on $\{|\mu| \star \delta_x |x \in K\}$. On the other hand the function $x \longrightarrow |\mu| \star \delta_x$ is weak*-continuous [6] into $M(X)$, so it is τ-continous when regarded as a map form K into $M(X)$ for each compact set K. as X is locally compact, the funciton is τ-continous on X. This completes the proof.

A Banach space A is said to have the *Dunford - Pettis* property if for each Banach space B and each W-compact operator T from A to B, $T(K)$ is compact in B whenever K is W-compact in A[7]. It is well known that $M(X)$ possesses this property [12].

Theorem.1.12.

Let $\mu \in L(X)$. Then the operator $R_\mu: M(K) \longrightarrow M(X)$ defined by $R_\mu(\nu)=\mu \star \nu$ is a W-compact mapping, where K is a compact subset of X.

Proof.

. By lemma 1.9 the set $\{\mu \star \delta_x | x \in K\}$ is weakly compact. Thus $W - \overline{CO}\ \{\mu \star \delta_x | x \in K\}$, the weak-closed convex hull of $\{\mu \star \delta_x | x \in K\}$ is weak-compact. Also on this set the weak and weak*-toplogies coinside. Now, we know that the closed-convex hull of $\{c\delta_x \mid |c|\ x \in K\}$ is the unit ball of $M(X)$ [6], also R_μ is weak*-continuous. Therefore, the image under R_μ of the unit ball of $M(K)$ is just $W - \overline{CO}\ \{\mu \star \delta_x | x \in K\}$ wich is weakly compact.

Lemma 1.13.

Let $\nu_1, \nu_2 \in L(X)$, and K be compact. Then $\{\nu_1 \star \nu_2 \star \delta_x | x \in K\}$ is norm-compact.

Proof.

Since the set of measures with compact support is dense in $L(X)$, we may assume without loss of generality that $supp\nu_2 = F$ is compact thus $F \star K$ is compact and $R_{\nu_2}: M(K) \longrightarrow M(F \star K)$, $R_{\nu_2}: M(F \star K) \longrightarrow M(X)$ are weakly - compact operators. Therefore $K_1 = \{\nu_2 \star \delta_x \mid x \in K\}$ is weakly compact and thus, by the Dunford-Pettis property $R_{\nu_1}(K_2) = \{\nu_1 \star \nu_2 \star \delta_x \mid x \in K\}$ is norm-compact.

Now, We establish a main result.

Theorem 1.14.

Let $\mu \in L(X)$. Then the function $x \longrightarrow \mu \star \delta_x$ is norm continuous.

Proof.

It is clear from 1.11 and 1.12 that $\overline{L(X)\star L(X)} \subseteq L_N(X)$ the space of all measures μ such that $x \longrightarrow \mu\star\delta_x$ is norm-continuous. Since $L(X)$ has a bounded approximate identity, we have $L(X)=\overline{L(X) \star L(X)} \subseteq L_N(X)$, it is also obvious that $L_N(X)\subseteq L(X)$. Thus $L(X)=L_N(X)$,i,e. $x \longrightarrow \mu\star\delta_x$ is norm-continuous.

Finally, we mention a result wich has an exact parallel in the semigroup case.

Propostion 1.15.

The algebra $L(X)$ has an identity if and only if X is discrete, in which case the identity element of $L(X)$ is δ_e.

We have also to mention that there is a one to one correspondence between the characters and complex homomorphisms of semigroup algebras. A modification of the proof 3.1 of [1] shows that there is a similar relation for $L(X)$, namely for each character ϕ there is a complex homorphism h such that $\phi(x)=\frac{h(\mu\star\delta_x)}{h(\mu)}$ where $\mu \in L(X)$, $h(\mu)\neq 0$.

It is true that hypergroups are the extesion of semigroup or group algebras to this type of algebras over locally compact spaces. However we also consider some different type of algbras with different approaches. For this we wait until the end of section 2.

2. Multipliers and isomorphisms.

This section is devoted to demonstrating results for $L(X)$ similar to those for group algebras. First of all it should be noted that, since $L(X)$ has an approximate identity of norm 1, $M(L(X))$ the multiplier algebra of $L(X)$ is isometrically isomorphic to $M(X)$ via the correspondence : $\mu\in L(X)$ corresponds to the multiplier T if and only if $T\nu=\nu\star\mu$.

A multipleir is called unitary if T is onto and an isometry. First, we characterize the unitary multipliers on $L(X)$. Now, we need some elementary results.

Lemma 2.1.

Let $(\nu_\alpha)_\alpha$ be a bounded net in $L(X)$ which tends to ν in the strong operator topology,*(i. e.* $\nu_\alpha\star\mu \longrightarrow \nu\star\mu$ for ever $\mu\in L(X)$). Then $(\nu_\alpha)_\alpha$ tends to ν in the weak*-topology $\sigma(M(X), C_o(X))$.

The proof is not too difficult so it is omitted.

Lemma 2.2.

Let $SE_x = \{ k\delta_x \mid x\in X, \mid k \mid = 1 \}$. Then $\overline{CO}\ [SE_x, SO] = \overline{CO}\ [SE_x, \sigma] =$ the unit ball of $M(X)$.

Note that we mean by $\overline{CO}[SE_X, SO]$ the closed convex hull of SE_X with strong operator toplogy, and the latter is the same with weak*-topology.

Lemma 2.3.

Let T be a unitary muliplier on $L(X)$. Then it can be extended to a unitary multiplier on $M(X)$.

Proof.

Because T is a unitary multiplier on $L(X)$ it has an inverse T^{-1} wich is also a unitary multiplier. Thus, there are measures τ, $\tau_1 \in M(X)$ such that $T\mu=\tau\star\mu$, $T^{-1}\mu=\tau_1\star\mu$ for $\mu\in L(X)$. Then $\delta_e\star\mu = \mu = T^{-1}T\mu$, so that $\tau_1\star\tau=\delta_e$, and similary, $\tau\star\tau_1=\delta_e$. Also $\|\tau\| = \|\tau_1\| = 1$. Define T on $M(X)$ by $\overline{T}$ on $M(X)$ by $\overline{T}\nu = \tau\star\nu$. Then $\overline{T}$ has an inverse, because $S\nu = \tau_1\star\nu$ is clearly an inverse to T and is an isometry because $\|\nu\| = \|\tau_1\star\tau\star\nu\| \leq \|\tau\star\nu\| \leq \|\nu\|$ $\nu\in M(X)$ for $\nu\in M(X)$. Hence T is unitary on $M(X)$.

Theorem 2.4.

Let T be a unitary multiplier on $L(X)$. Then there exists a constant $k(|k| = 1)$ and a homeomorphism α on X such that

$$T\mu(\psi) = k\mu(\psi o\alpha) \ (\psi\in C_o(X), \mu\in L(X))) \tag{1}$$

and

$$\lambda_{(x,y)}(\psi o\alpha) = \lambda_{(\alpha(x),y)}(\psi). \tag{2}$$

Conversely, if (2) holds then (1) defines a unitary multiplier T on $L(X)$.

Proof.

Since $\|T(\delta_x)\| = \|\delta_x\| = 1$, $T(\delta_x)$ is an extreme point of the unit ball of $M(X)$, also we have by lemma 2.3, $T(\delta_x) = \delta_x\star\tau$. Thus $T(\delta_x) = \delta_x\star\tau = k(x)\delta_{\alpha(x)}$ where k is a complex function on X with $(|k|=1)$ and α is a function from X to X. Therefore,

$$T(\delta_x)=T(\delta_x\star\delta_e)=T(\delta_e)\star\delta_x \tag{3}$$

$$= k(e)\delta_{\alpha(e)}\star\delta_x = k(e)\lambda_{(\alpha(e),x)} \tag{4}$$

So

$$k(x)\delta_{\alpha(x)} = k(e)\lambda_{(\alpha(e),x)}$$

Now, by integration over X we obtain $k(x) = k(e)$, $x\in X$. Thus $x \longrightarrow k(x)$ is a constant k, say. In the same way, T^{-1} gives rise to a function β, and it is easy to see that $\beta = \alpha^{-1}$. Since

$$x \longrightarrow \delta_x\star\tau=k\delta_{\alpha(x)}$$

is weakly continuous, α is continous. Similarly, α^{-1} is continuous, so that α is a homeomorphism. Because T is weak*-continous and linear on $M(X)$, we may extend the formula $T(\delta_x)(\psi) = k\delta_{\alpha(x)}(\psi) = k\delta_{(x)}(\psi o\alpha)$ $(\psi\in C_o(X))$ from measures of the form δ_x to every μ in $M(X)$ to obtain formula (1). The formula (2) comes from the relationship $T(\delta_x\star\delta_y)(\psi) = T(\delta_x)\star\delta_y(\psi)$ and (1).

Note 1.

We also have $T\mu = T(\delta_x\star\mu) = k\,\delta_{\alpha(e)}\star\mu$. This is a generalization of wendel's theorem [14]. It also shows that for every unitary multiplier T, there is a point $x\in X$ such that $T\mu = k\,\delta_\alpha\star\mu$ and it is easy to observe that the point is uniquely determind by $\alpha(e)$.

Note 2.

it can be shown that $I(L(X)$, the group of isometric multipliers on the commutative algebra $L(X)$ is a topological goup in the so-topology, when X is compact we have the follwoing.

Theorem 2.5.

If X is compact, then $I(L(X)$ is compat in the strong operator toplogy.

...st show that for each $\mu \in L(X)$, the set ... $\mid T \in I(L(X))\}$ is compact. But $B =$ ... $\mid |k| = 1$, is a homeo. on $X\}$. Since the ... $\longrightarrow k\mu\delta_x$ is contintuous, $\{k\ \mu{\star}\delta_{\alpha(e)} \mid \alpha$ ...eo. on $X\}$ is compact. Now as the set ... $1\}$ is compact and $x \longrightarrow \mu{\star}\delta_x \longrightarrow k\mu{\star}\delta_x$ ...us, $\{k\mu{\star}\delta_{x(e)} | \alpha$ is a homeo.$\}$ is compact. ...$(X))$ is compact in the strong operator ...This completes the proof of the theorem.

...we have

$$\ldots (\alpha(t))d\lambda_{(x,y)}(t)=\int \psi(t)\ d\lambda_{(\alpha(x),y)}(t)$$

...putting $x=e$, we get

$$\lambda_{(\alpha(x),y)} = \delta_{\alpha(y)}.$$

...fore $\alpha(e) \in Z(X)$ where $Z(X)$, the center ...he set $\{x \in X \mid supp\lambda_{(x,y)}$ is a single point ... $y \in X\}$. It follows that the group of ...ultipliers is contained in $Z(X)$.

...ples.

...s section we consider some examples. It ...e noted that the most famous examples of ...ps are the coset space and double coset ... a locally compact group and there are ...er examples in [9].

... we shall consider some other examples.

... $X = [0,1]$, and define $xoy = min\{x+y, 1\}$, ... is a foundation semigroup and $L(X) =$ ... $\oplus K\delta_1 \mid K \in C\}$, thus with $\lambda_{(x,y)} = \delta_{xoy}$ is ...dation hypergoup. Infact $L(X)$ is the ...on of the quotient space $L^1[0,1]$ with the hypergroup structure with $\lambda_{(x,y)}=\delta_{x+y}$ if $x+y<1$, $\lambda_{(x,y)}=0$ otherwise. The unitary multiplier group is just the unit circle.

II. In [6] another type of convolution was defined on a locally compact space X, namely, let $\psi \in C_o(X)$ and define

$$\mu{\star}\nu(\psi) = \iint T\psi_{(x,y)} d\mu(x) d\nu(y)$$

where $T\psi \in C_b(X \times X)$. With our condition on continuity, if we define $\lambda_{(x,y)}(\psi) = T\psi_{(x,y)}$ $(x,y \in X)$, with the additional condition that $T\psi_{(x,e)} = \psi(x)$ for some element e of X, then X has a hypergroup structure. With this modification the remainging property of [6] is still valid. More preciesly, if T is a lattice isomorphism on compact space X, and $\lambda_{(x,y)}$ is defined as before, then there exists a continuous map m of $X \times X$ into X and μ such that $\lambda_{(x,y)}(\psi) = \psi(m(x,y))\mu(x,y)$ where μ is a stricltly positive function on $X \times X$. If we further assume this multiplication is associative, we conclude that

$$\psi(m(x,m(y,z))) = \psi(m(m(x,y)z))$$

for all $\nu \in C(X)$. Thus

$$m(x,m(y,z)) = m(m(x,y)z).$$

That is $(x,y) \longrightarrow m(x,y)$ is an associative multiplication which makes X a compact semigroup. The group of unitary multipliers on X is isomorphic to $C \times G$ where C is the unit circle and G is the gorup generated by translations on X.

III. It is clear that if X is a locally compact semigroup with continuous multiplication, then $L(X) = L^1(G) \oplus \{K\delta_e \mid K \in C\}$, thus every commutative locally compact goup is a foundation hypergroup.

IV. In this final cxample we adapt an intersting example of [4] which is very suitable to our purpose in a very general sense. For notation we refer to [4], however, we explain very briefly some essentail points. Let $I=D^2\text{-}q$ be the Sturm-Lioville operator acting on R and q be a bounded variation function such that the function $p(x)=(1+|x|)q(x)$ is integrable on R. Define $M_w(R)$ to be the Banach space of measures on R with norm $\|\mu\|_\omega = \int_R w(t)d\,|\mu|(t)$, where w is a positive continuous increasing funtion on $[0, \infty]$. For each ψ, $T\psi$ is the unique solution of the system

$$(L\oplus I)u = (I\oplus L)u,$$

$$u(x,0) = \psi(x), (\frac{\partial u}{\partial y})(x,0) = B\psi(x)$$

whene I is an identity and B is the given operator which defines the boundary conditions. Since $T\psi$ is the unique solution of the above equation, $T\psi(x,0) = \psi(x)$. We put the additional restriction $T\psi>0$ whenever $\psi>0$. Now we define $\lambda_{(x,y)}(\psi)=T\psi(x,y)$ and the convoltution is defined by $\mu * \nu(\psi) = (\mu\oplus\nu)(T\psi)$. Thus R has a hypergroup structure with identity zero and weight norm defined by $\|\mu\| = \int \omega(t)d|\mu|(t)$.

The algebra $L(R)$ is $\psi^1(R)\oplus\{k\delta_o|k\in C\}$ we[...] $\psi^1(R)$ consists of the absolutely continous me[...]ures in $M_\omega(R)$. By our Theorem $L(R)$ has a bou[...]ded approximate identity and in this case $L(R)$ i[...] semi-simple algebra.

V. The most interesting hypergroups ha[...] invariant measures [3]. In this case $L(X)$ is j[...] $L^1(X,m)\oplus\{k\delta_e|k\in C\}$ where e is the identity of [...] Thus all of our results carry over to $L^1(X,m)$. T[...] interesting question is : Under what conditio[...] $L^1(X,m)$ "is" a goup algebra?

In [9] we have studied the second dual alge[...] of a hypergoup, and in other paper we ha[...] developed some isomorphism-problems on $L(X)$[...] the second dual of $L(X)$, which has not be[...] appeared so far.

References

[1] Baker Anne C. & J. W. "Algebras [...] measures on locally compact semigroups" III, [...] London Math. Soc., (2), 4(1972), 685-695.

[2] Bonsall F.F. & Duncan J., "Comple[...] Normed Algebras", Springer Varlag, Berli[...] Heidelberg, New York, 1973.

[3] Dunkl C.F., "The measure algebra of [...] locally compact hypergroup" Trans. Amer. mat[...] Soc., 179 (1973), 331-384.

[4] Ghahramani F. & Medgalchi A.R., Compa[...]

nultipliers on weightd hypergoup algebras, Math. Proc. Camb. Soc. 98, (1985), 493-500.

[5] Hutson V. & Pym J. S., "Measure algebra ssociated with a second order differential oper-tor" J. Functional analysis (1), 12 (1973), 68-96.

[6] Jewett R. I. "Spaces with an absract nvolution of measure" Advances in Math. 18 975), 1-101.

[7] Pym J.S. "Weakly separately continous easure algebras", Math Annalen 175 (1966), 7-219.

[8] A.R. Medghalchi, Isomorphisms and ltipiers on compact P_*-Hypergroups, Bull. nian math. Soc. Vol. 11, No, 182, PP, 33-37.

9] A.R. Medghalchi, the second dual algebra of ypergroup, Math. Z. 210, 615-624 (1992).

[10] Ross K.A., "Hypergroups and centers of measure algebras" symposia mathematica 22 (1977), 189-203. Institute Nazional di Alta Mathematica, Roma, Marzo 1976. London, New York 1977.

[11] Schaefer H. H. "Bananch Lattices and Positive Operators", Springer Verlag, Berlin, Heidlberg, New York 1974.

[12] __________ "Topological Vector Spaces" Macmillan, New York 1966.

[13] Spector, R. "Sur la structur locale des groups abelians localment compacts", Soc. Math. Frace Memoire 24 (1970), 1-94.

[14] Wendel J.W. "Left centralizers and isomorphisms of group algebras" Pacific J. Math. 2 (1952), 251-261.